SOLUTION TO PROBLEM #11902

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Problem #11902. Proposed by Cornel Ioan Valean, Teremia Mare, Timis, Romania. Let $\{x\}$ denote $x - \lfloor x \rfloor$, the fractional part of x. Prove

$$\int_0^1\int_0^1\int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^2 dx\,dy\,dz = 1 - \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2} + \frac{7\zeta(6)}{48} + \frac{\zeta(2)\zeta(3)}{18} + \frac{\zeta(3)^2}{18} + \frac{\zeta(3)\zeta(4)}{12}.$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. The 6 permutations of the chamber $0 \le x \le y \le z \le 1$ do partition the unit cube into tetrahedra. In view of the cyclic symmetry of the integrand, we obtain exactly 2 different triple integral evaluations, each appearing thrice.

Case 1 is modeled by $0 \le x \le y \le z \le 1$ and $\{\frac{x}{y}\}\{\frac{y}{z}\}\{\frac{z}{x}\} = \frac{x}{y}\frac{y}{z}\{\frac{z}{x}\} = \frac{x}{z}\{\frac{z}{x}\}$. Integrate by parts: let $u = \int_0^y(\cdots)$ and $v' = 1$, so $u' = \frac{y^2}{z^2}\{\frac{z}{y}\}^2 dy$ and $v = y$. Consequently,

$$\begin{aligned} I_1 &:= \int_0^1\int_0^z\int_0^y \left(\frac{x}{z}\left\{\frac{z}{x}\right\}\right)^2 dx\,dy\,dz = \int_0^1 \left(y\int_0^y \frac{x^2}{z^2}\left\{\frac{z}{x}\right\}^2 dx\right)_{y=0}^{y=z} dz - \int_0^1\int_0^z \frac{y^3}{z^2}\left\{\frac{z}{y}\right\}^2 dy\,dz \\ &= \int_0^1 z\int_0^z \frac{x^2}{z^2}\left\{\frac{z}{x}\right\}^2 dx\,dz - \int_0^1 z\int_0^z \frac{y^3}{z^3}\left\{\frac{z}{y}\right\}^2 dy\,dz. \end{aligned}$$

For the first integral, denoted by $I_{1,1}$, make the substitution $w = \frac{z}{x}$ so that $dx = -\frac{z\,dw}{w^2}$ and hence

$$\begin{aligned} I_{1,1} &= \int_0^1 z^2 dz \int_1^\infty \frac{\{w\}^2 dw}{w^4} = \frac{1}{3}\int_1^\infty \frac{\{w\}^2 dw}{w^4} = \frac{1}{3}\sum_{n=1}^\infty \int_0^1 \frac{\{\theta+n\}^2 d\theta}{(\theta+n)^4} = \frac{1}{3}\sum_{n=1}^\infty\int_0^1 \frac{\theta^2 d\theta}{(\theta+n)^4} \\ &= \frac{1}{9}\sum_{n=1}^\infty \frac{1}{n(1+n)^3} = \frac{1}{9}[3-\zeta(2)-\zeta(3)] = \frac{1}{3} - \frac{\zeta(2)}{9} - \frac{\zeta(3)}{9}. \end{aligned}$$

Likewise, the second integral results in $I_{1,2} := \int_0^1 z\int_0^z \frac{y^3}{z^3}\{\frac{z}{y}\}^2 dy\,dz = \frac{1}{6} - \frac{\zeta(3)}{18} - \frac{\zeta(4)}{12}$.

Case 2 is modeled by $0 \le x \le z \le y \le 1$ so that $\{\frac{x}{y}\}\{\frac{y}{z}\}\{\frac{z}{x}\} = \frac{x}{y}\{\frac{y}{z}\}\{\frac{z}{x}\}$. Make the substitution $w = \frac{z}{x}$ so that $dx = -\frac{z\,dw}{w^2}$ (analogously, $q = \frac{y}{z}, dz = -\frac{y\,dq}{q^2}$) and hence

$$\begin{aligned} I_2 &:= \int_0^1\int_0^y \left\{\frac{y}{z}\right\}^2 \int_0^z \left(\frac{x}{y}\left\{\frac{z}{x}\right\}\right)^2 dx\,dz\,dy = \int_0^1\int_0^y \left\{\frac{y}{z}\right\}^2 \frac{z^3}{y^2} dz\,dy \cdot \int_1^\infty \frac{\{w\}^2 dw}{w^4} \\ &= \int_0^1 y^2 dy \cdot \int_1^\infty \frac{\{q\}^2 dq}{q^5} \cdot \int_1^\infty \frac{\{w\}^2 dw}{w^4} = \frac{1}{3}\left(\frac{1}{2} - \frac{\zeta(3)}{6} - \frac{\zeta(4)}{4}\right)\left(1 - \frac{\zeta(2)}{3} - \frac{\zeta(3)}{3}\right). \end{aligned}$$

Combining all that we have found, the integral in the problem becomes $I = 3I_{1,1} - 3I_{1,2} + 3I_2$, i.e.

$$I = 1 - \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2} + \frac{\zeta(2)\zeta(4)}{12} + \frac{\zeta(2)\zeta(3)}{18} + \frac{\zeta(3)^2}{18} + \frac{\zeta(3)\zeta(4)}{12}.$$

The proof is complete. □

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