SOLUTION TO PROBLEM #11906

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Problem #11906. Proposed by Robert Bosch, Archimedean Academy, FL. Let x, y, and z be positive numbers such that $xyz = 1$. Prove

$$\sqrt{\frac{x+1}{x^2-x+1}} + \sqrt{\frac{y+1}{y^2-y+1}} + \sqrt{\frac{z+1}{z^2-z+1}} \leq 3\sqrt{2}.$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Consider $f(x) = \frac{x+1}{x^2-x+1} + \log(x) - 2$. We find that $f'(x) = \frac{(x-1)(x^3-2x^2-x-1)}{x(x^2-x+1)^2} = 0$ has exactly two real solutions for $x > 0$, because $g(x) = x^3 - 2x^2 - x - 1$ has a unique positive real root according to Descartes' Rule of Signs. These roots are: $r_1 = 1$ and another $1 < r_2 \leq 3$. Therefore $\max\{f(x) : 0 < x \leq 3\} = \max\{f(1), f(3)\} = 0$. If $x, y, z < 3$ and since $xyz = 1$, we obtain

$$\sum_{cyc} \frac{x+1}{x^2-x+1} \leq \sum_{cyc} (2 - \log x) = 6.$$

Suppose one of the numbers, WLOG say, $x \geq 3$. In this range, the function $g(x) = \frac{x+1}{x^2-x+1}$ is decreasing. Also $38(a^2-a+1) - 14(a+1) = 38a^2 - 52x + 38 > 0$ when $a > 0$. It follows that

$$\sum_{cyc} \frac{x+1}{x^2-x+1} \leq \frac{3+1}{3^2-3+1} + 2 \cdot \frac{38}{14} = 6.$$

In all cases, $\sum_{cyc} \frac{x+1}{x^2-x+1} \leq 6$. Cauchy-Schwarz $Aa + Bb + Cc \leq \sqrt{(A^2+B^2+C^2)(a^2+b^2+c^2)}$ with $A = B = C = 1, a = \sqrt{\frac{x+1}{x^2-x+1}}, b = \sqrt{\frac{y+1}{y^2-y+1}}$, and $c = \sqrt{\frac{z+1}{z^2-z+1}}$ completes the argument

$$\sum_{cyc} \sqrt{\frac{x+1}{x^2-x+1}} \leq \sqrt{3 \sum_{cyc} \frac{x+1}{x^2-x+1}} \leq \sqrt{3 \cdot 6} = 3\sqrt{2}. \qquad \square$$

Notation. Here, $\sum_{cyc} g(x)$ means the cyclic sum $g(x) + g(y) + g(z)$.

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