SOLUTION TO PROBLEM #11909

Problem #11909. Proposed by H. Ohtsuka, Japan. Prove that for every positive integer m, there exists a polynomial P_m in two variables, with integer coefficients, such that for all integers n and r with $0 \leq r \leq n$,

$$\sum_{k=-r}^{r} \binom{n}{r+k}\binom{n}{r-k} k^{2m} = \frac{P_m(n,r)}{\prod_{j=1}^{m}(2n-2j+1)}\binom{2n}{2r}.$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. From the Vandermonde-Chu identity, we get $\sum_{k=-r}^{r}\binom{n}{r+k}\binom{n}{r-k} = \sum_{i=0}^{2r}\binom{n}{i}\binom{n}{2r-i} = \binom{2n}{2r}$. Next, observe that k^{2m} can be expanded in a different bases as follows

$$k^{2m} = \sum_{i=0}^{m}\alpha_i \prod_{t=0}^{i-1}(r+k-t)(r-k-t) = \sum_{i=0}^{m}\alpha_i \frac{(r+k)!(r-k)!}{(r+k-i)!(r-k-i)!} \qquad \text{for some } \alpha_i(r) \in \mathbb{Z}[r].$$

For example, $k^2 = -(r+k)(r-k) + r^2$. Then, the sum on the left-hand side of the original problem takes the form

$$\begin{aligned}\sum_{i=0}^{m}\sum_{k=-r}^{r}\binom{n}{r+k}\binom{n}{r-k}\frac{\alpha_i\,(r+k)!(r-k)!}{(r-i+k)!(r-i-k)!} &= \sum_{i=0}^{m}\frac{\alpha_i\, n!^2}{(n-i)!^2}\sum_{k=-(r-i)}^{r-i}\binom{n-i}{r-i+k}\binom{n-i}{r-i-k} \\ &= \sum_{i=0}^{m}\frac{\alpha_i\, n!^2}{(n-i)!^2}\binom{2n-2i}{2r-2i} \\ &= \binom{2n}{2r}\sum_{i=0}^{m}\alpha_i \frac{n!^2}{(n-i)!^2}\frac{(2r)!}{(2r-2i)!}\frac{(2n-2i)!}{(2n)!}.\end{aligned}$$

If $1 \leq i \leq m$, then $\frac{n!(n-1)!}{(n-i)!^2}$ and $\frac{(2r-1)!\,r}{(2r-2i)!}$ are polynomials, in n and r respectively, of finite degree (depending on m). The remaining contributions simplify to $\frac{2n\cdot(2n-2i)!}{(2n)!} = \frac{1}{\prod_{j=1}^{i}(2n-2j+1)}$ is a rational function with denominator a factor of $\prod_{j=1}^{m}(2n-2j+1)$. The polynomial promised under the claim is then

$$P_m(n,r) = \sum_{i=0}^{m}\alpha_j(r)\frac{n!(n-1)!(2r-1)!\,r}{(n-i)!^2(2r-2i)!}\prod_{j=i+1}^{m}(2n-2j+1).$$

The proof is complete. $\square$