SOLUTION TO PROBLEM #11919

Problem #11919. *Proposed by Arkady Alt, San Jose, CA.* For positive integers m, n and k, with $k \geq 2$, prove

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} (\min\{i_1, \dots, i_k\})^m = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} ((n+1)^i - n^i) \sum_{j=1}^{n} j^{k+m-i}.$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, LA, USA. Start with the right-hand side to change the order of summation and apply the Binomial Theorem:

$$\begin{aligned} RHS &= \sum_{j=1}^{n} (-1)^m j^{k+m} \sum_{i=1}^{m} (-1)^i \binom{m}{i} \left[\left(\frac{n+1}{j} \right)^i - \left(\frac{n}{j} \right)^i \right] \\ &= \sum_{j=1}^{n} (-1)^m j^{k+m} \left[\left(1 - \frac{n+1}{j} \right)^m - \left(1 - \frac{n}{j} \right)^m \right] \\ &= \sum_{j=1}^{n} j^k \left[(n+1-j)^m - (n-j)^m \right]. \end{aligned}$$

Denote the LHS by $a_n(k;m)$ and adopt the convention $0^0 = 1$ and $a_n(0;m) = (n+1)^m$. Next, reduce dimensions according to how many of the entries among the k-tuple $(i_1, \dots, i_k)$ are equal to n. There are exactly $\binom{k}{\ell}$ ways to have ℓ of them. Therefore, $a_n(k;m) = \sum_{\ell=0}^{k} \binom{k}{\ell} a_{n-1}(\ell;m)$. The aim is to show $a_n(k;m) = \sum_{j=0}^{n} j^k \left[(n+1-j)^m - (n-j)^m \right]$, by induction on $n \geq 1$. Notice the (harmless) index $j = 0$. The case $n = 1$: $1^m = a_1(0;m) = \sum_{j=0}^{1} j^k \left[(1+1-j)^m - (1-j)^m \right] = 1$. So, assume the claim holds for integers $< n$. The relation $a_n(k;m) = \sum_{\ell=0}^{k} \binom{k}{\ell} a_{n-1}(\ell;m)$ implies

$$\begin{aligned} a_n(k;m) &= \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{j=0}^{n-1} j^\ell [(n-j)^m - (n-1-j)^m] \\ &= \sum_{j=0}^{n-1} [(n-j)^m - (n-1-j)^m] \sum_{\ell=0}^{k} \binom{k}{\ell} j^\ell \\ &= \sum_{j=0}^{n-1} [(n-j)^m - (n-1-j)^m] (j+1)^k \\ &= \sum_{j=1}^{n} [(n+1-j)^m - (n-j)^m] j^k, \end{aligned}$$

which confirms the validity of the assertion for n. The proof follows. □

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