SOLUTION TO PROBLEM #11920

Problem #11920. *Proposed by A. Plaza and S. Falcon, Spain.* For positive integer k, let $\{F_{k,n}\}_{n\geq 0}$ be the sequence defined by initial conditions $F_{k,0}=0, F_{k,1}=1$, and the recurrence $F_{k,n+1}=kF_{k,n}+F_{k,n-1}$. Find a closed formula for $\sum_{i=0}^{n}\binom{2n+1}{i}F_{k,2n+1-2i}$.

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, LA, USA. Binet's formula gives $F_{k,n}=\frac{\lambda_+^n-\lambda_-^n}{\lambda_+-\lambda_-}$ where $\lambda_\pm=\frac{k\pm\sqrt{k^2+4}}{2}$. Direct expansion using the Binomial Theorem, leads to $F_{k,n}=\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-j}{j}k^{n-1-2j}$. Thus, $F_{k,2n+1-2i}=\sum_{j=0}^{n-i}\binom{2n-2i-j}{j}k^{2n-2i-2j}$. After repeated reindexing, we obtain

$$\sum_{i=0}^{n}\binom{2n+1}{i}F_{k,2n+1-2i}=\sum_{i=0}^{n}\binom{2n+1}{n-i}\sum_{j=0}^{i}\binom{2i-j}{j}k^{2i-2j}=\sum_{i=0}^{n}\binom{2n+1}{n-i}\sum_{j=0}^{i}\binom{i+j}{2j}k^{2j}.$$

Let's swap summation $\sum_{i=0}^{n}\binom{2n+1}{n-i}\sum_{j=0}^{i}\binom{i+j}{2j}k^{2j}=\sum_{j=0}^{n}k^{2j}\sum_{i=j}^{n}\binom{2n+1}{n-i}\binom{i+j}{2j}$. Next, we show $f_j(n):=\sum_{i=j}^{n}\frac{\binom{2n+1}{n-i}\binom{i+j}{2j}}{4^{n-j}\binom{n}{j}}=1$ using the Wilf-Zeilberger method. Denote $F_j(n,i):=\frac{\binom{2n+1}{n-i}\binom{i+j}{2j}}{4^{n-j}\binom{n}{j}}$. This involves a routine procedure: if $G_j(n,k)=\frac{(j-i)F_j(n,i)}{2(n+1-i)}$ then verify $F_j(n+1,i)-F_j(n,i)=G_j(n,i+1)-G_j(n,i)$. Sum both sides over the integers to yield $f_j(n+1)-f_j(n)=0$, and check $f_j(0)=1$. These steps lead to the claim.

Put all these together with the Binomial Theorem: $\sum_{i=0}^{n}\binom{2n+1}{i}F_{k,2n+1-2i}=\sum_{j=0}^{n}k^{2j}4^{n-j}\binom{n}{j}=(k^2+4)^n$. Therefore, the required closed formula is $(k^2+4)^n$. $\square$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX