SOLUTION TO PROBLEM #11923

Problem #11923. Proposed by Orman Kouba, Damascus, Syria. Let f_p be the function on $(0, \frac{\pi}{2})$ given by

$$f_p(x) = (1+\sin x)^p - (1-\sin x)^p - 2\sin(px).$$

Prove $f_p > 0$ for $0 < p < \frac{1}{2}$ and $f_p < 0$ for $\frac{1}{2} < p < 1$.

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Using the bijective map $u = \sin x$ translate the function to

$$F_p(u) = \arcsin\left(\frac{(1+u)^p - (1-u)^p}{2}\right) - p\arcsin u, \qquad 0 < u < 1,$$

and the inequalities into: (a) $F_p > 0$ for $0 < p < \frac{1}{2}$; (b) $F_p(u) < 0$ for $\frac{1}{2} < p < 1$. Below, $g' = \frac{d}{du}g$.

(a): we prove $F_p' > 0$ when $0 < p < \frac{1}{2}$. Equivalently, we verify the inequalities

$$\begin{aligned}
& \frac{p}{2}\,\frac{(1+u)^{p-1} + (1-u)^{p-1}}{\sqrt{1-\frac{1}{4}[(1+u)^p-(1-u)^p]^2}} > \frac{p}{\sqrt{1-u^2}} \\
\iff \quad & (1-u^2)\left[(1+u)^{p-1} + (1-u)^{p-1}\right]^2 + [(1+u)^p - (1-u)^p]^2 > 4 \\
\iff \quad & \frac{(1+u)^{2p}(1-u^2)}{(1+u)^2} + \frac{(1-u)^{2p}(1-u^2)}{(1-u)^2} + (1+u)^{2p} + (1-u)^{2p} > 4 \\
\iff \quad & (1+u)^{2p-1} + (1-u)^{2p-1} > 2.
\end{aligned}$$

Since $0 < p < \frac{1}{2}, 0 < u < 1$, we apply AGM to justify the last inequality

$$(1+u)^{2p-1} + (1-u)^{2p-1} \geq \frac{2}{(1-u^2)^{\frac{1}{2}-p}} > 2.$$

Therefore, $F_p(u)$ is a strictly increasing over $0 < u < 1$. In particular, $F_p(u) > F_p(0) = 0$.

(b): we prove $F_p' < 0$ when $\frac{1}{2} < p < 1$. A similar argument as above reduces the claim to $H_p(u) := (1+u)^{2p-1} + (1-u)^{2p-1} - 2 < 0$. Since $\frac{1}{1+u} < \frac{1}{1-u}$ and $2p-2 < 0$, we argue that

$$H_p'(u) = (2p-1)[(1+u)^{2p-2} - (1-u)^{2p-2}] < 0.$$

Thus, H_p is strictly decreasing over $0 < u < 1$. In particular, $H_p(u) < H_p(0) = 0$. The assertion follows and the proof is complete. □