SOLUTION TO PROBLEM #11924

Problem #11924. Proposed by Cornel Ioan Valean, Timis, Romania. Calculate

$$\int_0^{\pi/2} \frac{\{\tan x\}}{\tan x} dx$$

where $\{u\}$ denotes $u - \lfloor u \rfloor$.

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA.

$$\int_0^{\pi/2} \frac{\{\tan x\}}{\tan x} dx = \int_0^{\pi/2} dx - \int_0^{\pi/2} \frac{\lfloor \tan x \rfloor}{\tan x} dx = \frac{\pi}{2} - \int_0^{\pi/2} \frac{\lfloor \tan x \rfloor}{\tan x} dx.$$

So, we focus on the latter integral. Substitute $y = \tan x$ so that

$$\begin{aligned}
\int_0^{\pi/2} \frac{\lfloor \tan x \rfloor}{\tan x} dx &= \int_0^{\infty} \frac{\lfloor y \rfloor \, dy}{y(1+y^2)} = \sum_{n=0}^{\infty} n \int_n^{n+1} \frac{dy}{y(1+y^2)} = \sum_{n=1}^{\infty} n \int_n^{n+1} \frac{dy}{y(1+y^2)} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[n \log\left(\frac{(1+n)^2}{1+(1+n)^2} \right) - n \log\left(\frac{n^2}{1+n^2} \right) \right] \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left[(n+1) \log\left(\frac{(1+n)^2}{1+(1+n)^2} \right) - n \log\left(\frac{n^2}{1+n^2} \right) \right] + \frac{1}{2} \sum_{n=2}^{\infty} \log\left(1 + \frac{1}{n^2} \right) \\
&= \frac{1}{2} \log 2 + \frac{1}{2} \sum_{n=2}^{\infty} \log\left(1 + \frac{1}{n^2} \right),
\end{aligned}$$

where we compute the first sum by telescoping. Recall the infinite product

$$\frac{\sinh z}{z} = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 n^2} \right).$$

Taking logarithms on both sides and replacing $z = \pi$ gives $\log\left(\frac{\sinh \pi}{\pi}\right) = \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n^2}\right)$. That means $\sum_{n=2}^{\infty} \log\left(1 + \frac{1}{n^2}\right) = \log\left(\frac{\sinh \pi}{\pi}\right) - \log 2$. Combining the above calculations, we arrive at the following evaluation

$$\int_0^{\pi/2} \frac{\{\tan x\}}{\tan x} dx = \frac{\pi}{2} - \frac{1}{2} \log\left(\frac{\sinh \pi}{\pi} \right). \qquad \square$$