SOLUTION TO PROBLEM #11925

Problem #11925. Proposed by Leonard Giugiuc, Romania. Let n be an integer with $n \geq 4$. Find the largest k such that for any list a of n real numbers that sum to 0,

$$\left(\sum_{j=1}^{n} a_j^2 \right)^3 \geq k \left(\sum_{j=1}^{n} a_j^3 \right)^2 .$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Just replace $n \to n+1$ for convenience. Denote the elementary symmetric functions by $e_m(a)$. Since $a_{n+1} = -\sum_{j=1}^{n} a_j$, $\sum_{j=1}^{n} a_j^2 = e_1^2 - 2e_2$ and $\sum_{j=1}^{n} a_j^3 = e_1^3 - 3e_1e_2 + 3e_3$, we have

$$\sum_{j=1}^{n+1} a_j^2 = \sum_{j=1}^{n} a_j^2 + \left(\sum_{j=1}^{n} a_j \right)^2 = e_1^2 - 2e_2 + e_1^2 = 2(e_1^2 - e_2)$$
$$\sum_{j=1}^{n+1} a_j^3 = \sum_{j=1}^{n} a_j^3 - \left(\sum_{j=1}^{n} a_j \right)^3 = e_1^3 - 3e_1e_2 + 3e_3 - e_1^3 = 3(e_3 - e_1e_2).$$

We claim $k = 3$, so the required inequality takes the form

$$8(e_1^2 - e_2)^3 \geq 27(e_1e_2 - e_3)^2 \tag{1}$$

where $e_m = e_m(a) = e_m(a_1, \ldots, a_n)$ and $a \in \mathbb{R}^n$ for $n \geq 3$.

The case $n = 3$. In three variables, a routine calculation shows

$$2e_1^2 - 2e_2 = (a_1 + a_2)^2 + (a_2 + a_3)^2 + (a_3 + a_1)^2$$
$$(e_1e_2 - e_3)^2 = (a_1^2a_2 + a_1a_2^2 + a_2^2a_3 + a_2a_3^2 + a_3^2a_1 + a_3a_1^2)^2 = (a_1 + a_2)^2(a_2 + a_3)^2(a_3 + a_1)^2.$$

Applying AGM, we find that

$$\begin{aligned} \left(\frac{2e_1^2 - 2e_3}{3} \right)^3 &= \left(\frac{(a_1 + a_2)^2 + (a_2 + a_3)^2 + (a_3 + a_1)^2}{3} \right)^3 \\ &\geq (a_1 + a_2)^2(a_2 + a_3)^2(a_3 + a_1)^2 \\ &= (e_1e_2 - e_3)^2 \end{aligned}$$

proves the desired inequality.