SOLUTION TO PROBLEM #11926

Problem #11926. Proposed by O. Furdui, Romania. Let k be an integer, $k \geq 2$. Find

$$I_k := \int_0^\infty \frac{\log|1-x|}{x^{1+1/k}} dx.$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Split up the integral into two parts and make the substitution $y = \frac{1}{x}$ on the second part to obtain

$$I_k = \int_0^1 \frac{\log(1-x)}{x^{1+1/k}} dx + \int_0^1 \frac{\log(1-y)}{y^{1-1/k}} dy - \int_0^1 \frac{\log y}{y^{1-\frac{1}{k}}} dy.$$

Recall Euler's beta function $B(u,v) = \int_0^1 (1-x)^{u-1} x^{v-1} dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$, with derivative

$$\frac{dB}{du}(1,v) = \frac{\Gamma(v)\Gamma(1)}{\Gamma(1+v)} \left[\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(1+v)}{\Gamma(1+v)} \right] = \frac{\Psi(1)-\Psi(1+v)}{v} = \int_0^1 x^{v-1} \log(1-x)\, dx;$$

where $\Psi(v) = \frac{\Gamma(v)}{\Gamma(v)}$ is the *digamma* function. Also, if we denote $F(z) := \int_0^1 y^z dy = \frac{1}{1+z}$ then $F'(z) = -\frac{1}{(1+z)^2} = \int_0^1 y^z \log y\, dy$. In light of these functions, we have

$$\begin{aligned} I_k &= \frac{\Psi(1)-\Psi(1-\frac{1}{k})}{-1/k} + \frac{\Psi(1)-\Psi(1+\frac{1}{k})}{1/k} - F'(-1+1/k) \\ &= k\left[\Psi\left(1-\frac{1}{k}\right) - \Psi\left(1+\frac{1}{k}\right)\right] + k^2 \\ &= k\left[\Psi\left(1-\frac{1}{k}\right) - \Psi\left(\frac{1}{k}\right) - k\right] + k^2 \\ &= k\left[\Psi\left(1-\frac{1}{k}\right) - \Psi\left(\frac{1}{k}\right)\right] \\ &= k\,\pi \cot\left(\frac{\pi}{k}\right); \end{aligned}$$

where in the last equality we used the *reflection formula* for Ψ inherited from the *Gamma* function. We conclude that $I_k = \pi k \cot(\pi/k)$. $\square$