SOLUTION TO PROBLEM #11928

Problem #11928. Proposed by H. Ohtsuka, Japan. For positive integers n and m and for a sequence $\{a_i\}_{i\geq 1}$, prove

$$(a) \qquad \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}a_{i+j}=\sum_{k=0}^{n+m}\binom{n+m}{k}a_k$$

and

$$(b) \qquad \sum_{0\leq i<j\leq n}\binom{n}{i}\binom{n}{j}\binom{i+j}{n}=\sum_{0\leq i<j\leq n}\binom{n}{i}\binom{n}{j}^2.$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. (a) follows from the Vandermonde-Chu identity after the substitution $i+j=k$:

$$\sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}a_{i+j}=\sum_{k=0}^{n+m}a_k\sum_{j=0}^{m}\binom{n}{k-j}\binom{m}{j}=\sum_{k=0}^{m+n}a_k\binom{n+m}{k}.$$

(b): due to symmetry in i and j, we have $\sum_{0\leq i<j\leq n}\binom{n}{i}\binom{n}{j}\binom{i+j}{n}=\sum_{0\leq j<i\leq n}\binom{n}{i}\binom{n}{j}\binom{i+j}{n}$. Also,

$$\sum_{j=0}^{n}\sum_{i=0}^{j-1}\binom{n}{i}\binom{n}{j}^2=\sum_{i=0}^{n}\sum_{j=i+1}^{n}\binom{n}{i}\binom{n}{n-j}^2=\sum_{i=0}^{n}\sum_{j=0}^{n-i-1}\binom{n}{n-i}\binom{n}{j}^2=\sum_{i=0}^{n}\sum_{j=0}^{i-1}\binom{n}{i}\binom{n}{j}^2$$

shows that $\sum_{0\leq i<j\leq n}\binom{n}{i}\binom{n}{j}^2=\sum_{0\leq j<i\leq n}\binom{n}{i}\binom{n}{j}^2$. Furthermore, on the diagonal (across $i=j$) we claim $f(n):=\sum_{j=0}^{n}\binom{n}{j}^2\binom{2j}{n}=\sum_{j=0}^{n}\binom{n}{j}^3:=g(n)$ holds. This is provable by Zeilberger's algorithm which verifies that both $f(n)$ and $g(n)$ satisfy the recurrence $y(0)=1, Y(1)=2$ together with

$$-8(n+1)^2Y(n)-(7n^2+21n+16)Y(n+1)+(n+2)^2Y(n+2)=0.$$

Thus, (b) would follow if we prove $F(n):=\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}\binom{i+j}{n}=\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}^2:=G(n)$. To this end, we proceed with $G(n)=\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{n}\binom{n}{j}^2=\sum_{i=0}^{n}\binom{n}{i}\binom{2n}{n}$ by Vandermonde-Chu. Next, applying part (a) with $a_{i+j}=\binom{i+j}{n}$ and changing indices results in

$$F(n)=\sum_{k=0}^{2n}\binom{2n}{k}\binom{k}{n}=\sum_{k=n}^{2n}\binom{2n}{k}\binom{k}{n}=\sum_{i=0}^{n}\binom{2n}{n+i}\binom{n+i}{n}=\sum_{i=0}^{n}\binom{2n}{n}\binom{n}{i}$$

where $\frac{(2n)!(n+i)!}{(n+i)!(n-i)!n!i!}=\frac{(2n)!}{(n-i)!n!i!}=\frac{(2n)!n!}{n!^2(n-i)!i!}$ has been used. So, $F(n)=G(n)$. This completes the proof of part (b) and the given problem. $\square$

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