SOLUTION TO PROBLEM #12022

Problem #11928. Proposed by Mircea Merca, University of Craiova, Craiova, Romania. Let n be a positive integer, and let x be a real number not equal to -1 or 1. Prove

$$\sum_{k=0}^{n-1} \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} = n \tag{1}$$

and

$$\sum_{k=0}^{n-1} (-1)^k \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} x^{\binom{n-1-k}{2}} = n x^{\binom{n}{2}}. \tag{2}$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. First, note that identity (2) becomes (1) after replacing $x \mapsto \frac{1}{x}$ and some algebraic simplification. Let $[x]_m = (1-x^m)\cdots(1-x)$ for $m \in \mathbb{N}$ and $[x]_0 = 1$. Denote the LHS of (1) by $f_n(x)$. Thus

$$\begin{aligned} f_{n-1} - f_n &= (1-x^n)\cdots(1-x) + \sum_{k=0}^{n-1} \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n+1-k})}{1-x^{k+1}} \left\{x^{n-k} - x^{n+1}\right\} \\ &= (1-x^n)\cdots(1-x) + \sum_{k=0}^{n-1} x^{n-k}(1-x^n)\cdots(1-x^{n+1-k}) \\ &= \sum_{k=0}^{n} x^{n-k} \frac{[x]_n}{[x]_{n-k}} = \sum_{k=0}^{n} x^k \frac{[x]_n}{[x]_k}. \end{aligned}$$

We induct on n to show $\sum_{k=0}^{n} \frac{x^k}{[x]_k} = \frac{1}{[x]_n}$. This is clear when $n = 0$. Assume it holds for n. It follows that $\sum_{k=0}^{n+1} \frac{x^k}{[x]_k} = \frac{x^{n+1}}{[x]_{n+1}} + \sum_{k=0}^{n} \frac{x^k}{[x]_k} = \frac{x^{n+1}}{[x]_{n+1}} + \frac{1}{[x]_n} = \frac{1}{[x]_n}\left(\frac{x^{n+1}}{1-x^{n+1}} + 1\right) = \frac{1}{[x]_{n+1}}$ justifies the claim. That means, $f_{n+1}(x) - f_n(x) = 1$. It is easy to check $f_1(x) = 1$. Therefore, $f_n(x) = n$ as desired. □