Problem 12091
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Prove

$$2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{i!j!k!}{ij(i+j+k)!}\left(H_{i+j+k}-H_k\right)=\zeta(3)$$

where H_k is the k-th harmonic number.

Solution proposed by Tewodros Amdeberhan, Department of Mathematics, Tulane University, New Orleans, USA, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Roma, Italy.

Solution. Suppressing x, let $F(i,j,k):=\frac{(i-1)!\,(j-1)!\,(x)_k}{(x)_{i+j+k}}$. Then, one can *routinely* check that

$$F(i,j,k+1)+F(i,j,k)=F(i,j,k)-F(i+1,j,k)+F(i,j,k)-F(i,j+1,k).$$

Sum this over all positive integers i and j and apply telescoping on the right-hand side so that

$$\sum_{i,j\geq 1}F(i,j,k+1)+\sum_{i,j\geq 1}F(i,j,k)=\sum_{j\geq 1}F(1,j,k)+\sum_{i\geq 1}F(i,1,k)=2\sum_{i\geq 1}F(i,1,k). \tag{1}$$

Now, let $G_1(i,k):=\frac{(i-1)!\,(x)_k}{(x)_{i+k}\,(x+k)}$ then $F(i,1,k).=G_1(i,k)-G_1(i+1,k)$ and we get

$$\sum_{i\geq 1}F(i,1,k)=\sum_{i\geq 1}(G_1(i,k)-G_1(i+1,k))=G_1(1,k)=\frac{1}{(x+k)^2}. \tag{2}$$

Combining (1) and (2), we have

$$\begin{aligned}2\sum_{i,j,k\geq 1}F(i,j,k)&=\sum_{i,j\geq 1}F(i,j,1)+\sum_{i,j,k\geq 1}F(i,j,k+1)+\sum_{i,j,k\geq 1}F(i,j,k)\\&=f(x)+2\sum_{i,k\geq 1}F(i,1,k)=f(x)+2\sum_{k\geq 1}\frac{1}{(x+k)^2}\end{aligned}$$

where

$$f(x):=\sum_{i,j\geq 1}F(i,j,1)=\sum_{i\geq 1}\sum_{j\geq 1}(G_2(i,j)-G_2(i,j+1))=\sum_{i\geq 1}G_2(i,1)=\sum_{i\geq 1}\frac{(i-1)!}{(x+1)_i\,(x+i)}$$

with $G_2(i,j):=\frac{(i-1)!(j-1)!}{(x+1)_{i+j-1}\,(x+i)}$ and $F(i,j,1)=G_2(i,j)-G_2(i,j+1)$. Note that

$$\begin{aligned}f'(1)&=\sum_{i\geq 1}\frac{1}{i\,(i+1)^2}\left(-H_i+1-\frac{2}{i+1}\right)\\&=\sum_{i\geq 1}\left(\frac{2}{(i+1)^3}+\frac{H_i}{(i+1)^2}+\frac{1}{i+1}-\frac{1}{i}+\frac{1}{(i+1)^2}-\frac{1}{i^2}+\frac{H_i}{i+1}-\frac{H_{i-1}}{i}\right)\\&=2(\zeta(3)-1)+\zeta(3)-1-1=3\zeta(3)-4.\end{aligned}$$

Consequently,

$$\begin{aligned}2\sum_{i,j,k\geq 1}\frac{i!\,j!\,k!\,(H_{i+j+k}-H_k)}{i\,j\,(i+j+k)!}&=-\frac{d}{dx}\left(2\sum_{i,j,k\geq 1}F(i,j,k)\right)\Bigg|_{x=1}\\&=-f'(1)+4\sum_{k\geq 1}\frac{1}{(1+k)^3}\\&=-3\zeta(3)+4+4(\zeta(3)-1)=\zeta(3).\end{aligned}$$

□