SOLUTION TO PROBLEM #12099

Problem #12099. Proposed by M. Bataille. Let m and n be integers with $0 \le m \le n-1$. Evaluate

$$\sum_{k=0,k\neq m}^{n-1} \cot^2\left(\frac{(m-k)\pi}{n}\right).$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Without loss of generality, we may assume $m=0$ since any other value of m simply renames the same summands. If $f(r)$ is periodic in $\mathbb{Z}$ of period n and $\xi = e^{\frac{2\pi i}{n}}$, then the *discrete Fourier transform* is

$$f(r) = \frac{1}{n}\sum_{k=0}^{n-1} \hat{f}(j)\xi^{rk} \qquad \text{where} \qquad \hat{f}(k) = \sum_{r=0}^{n-1} f(r)\xi^{-rk}.$$

Let $f(r) = \left\{\frac{r}{n}\right\} - \frac{1}{2}$ when $n \not| r$ and $f(r)=0$ otherwise; here $\{x\} = x - \lfloor x \rfloor$. Noting $\xi^n = 1$, we have

$$\hat{f}(k) = \sum_{r=1}^{n-1}\left(\frac{r}{n}-\frac{1}{2}\right)\xi^{-rk} = \frac{1}{n}\sum_{r=1}^{n-1} r\xi^{-rk} - \frac{1}{2}\sum_{r=1}^{n-1}\xi^{-rk} = \frac{\xi^k}{1-\xi^k} + \frac{1}{2} = \frac{1+\xi^k}{2(1-\xi^k)} = \frac{i}{2}\cot\left(\frac{k\pi}{n}\right).$$

Using the *convolution* formula $\frac{1}{n}\sum_{k=1}^{n-1}\hat{f}(k)\hat{f}(-k) = \sum_{r=1}^{n-1} f(r)f(r)$, we obtain

$$\begin{aligned}\frac{1}{4n}\sum_{k=1}^{n-1}\cot^2\left(\frac{k\pi}{n}\right) &= -\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{i}{2}\cot\left(\frac{k\pi}{n}\right)\right)^2 = \sum_{r=1}^{n-1}\left(\frac{r}{n}-\frac{1}{2}\right)^2 = \frac{1}{n^2}\sum_{r=1}^{n-1} r^2 - \frac{1}{n}\sum_{r=1}^{n-1} r + \sum_{r=1}^{n-1}\frac{1}{4} \\ &= \frac{(n-1)n(2n-1)}{6n^2} - \frac{(n-1)n}{2n} + \frac{n-1}{4} = \frac{(n-1)(n-2)}{12n}.\end{aligned}$$

Therefore, we conclude that

$$\sum_{k=0,k\neq m}^{n-1}\cot^2\left(\frac{(m-k)\pi}{n}\right) = \frac{(n-1)(n-2)}{3}. \qquad \square$$