SOLUTION TO PROBLEM #12106

Problem #12106. Proposed by H. Ohtsuka, Japan. For any positive integer n, prove

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{ij}=\sum_{1\le i\le j\le n}\frac{2^n-2^{n-i}}{ij}.$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Denote the LHS and RHS by f_n and g_n, respectively, and induct on n. It's obvious $f_1=g_1$. Assume $f_n=g_n$. For the case $n+1$, proceed with $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$ and split the sum

$$\begin{aligned}
f_{n+1}&=\sum_{k=1}^{n+1}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{ij}+\sum_{k=1}^{n+1}\binom{n}{k-1}\sum_{1\le i\le j\le k}\frac{1}{ij}=f_n+1+\sum_{k=1}^{n}\binom{n}{k}\sum_{1\le i\le j\le k+1}\frac{1}{ij}\\
&=2f_n+1+\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{i=1}^{k+1}\frac{1}{i}=2f_n+\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{i=1}^{k+1}\frac{1}{i}\\
&=2f_n+\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\int_0^1\frac{1-x^k}{1-x}\,dx=2f_n+\frac{1}{n+1}\int_0^1\frac{dx}{1-x}\sum_{k=0}^{n}\binom{n+1}{k+1}(1-x^{k+1})\\
&=2f_n+\frac{1}{n+1}\int_0^1\frac{2^{n+1}-(1+x)^{n+1}}{1-x}\,dx=2f_n+\frac{1}{n+1}\int_0^1\sum_{i=1}^{n+1}2^{n+1-i}(1+x)^{i-1}\,dx\\
&=2f_n+\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{2^{n+1}-2^{n+1-i}}{i}.
\end{aligned}$$

On the other hand, it is easy to see $g_{n+1}=2g_n+\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{2^{n+1}-2^{n+1-i}}{i}$. The proof follows. □

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX