SOLUTION TO PROBLEM #12107

Problem #12107. Proposed by c. I. Valean, Romania. Prove

$$\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{1+x^2}\,\sqrt{1+y^2}\,(1-x^2y^2)} = G$$

where G is the Catalan's constant $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$.

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Start with the substitution $x=\tan\theta$ and $y=\tan\beta$ so that the given integral transforms into

$$\begin{aligned}
\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\sec\theta\,\sec\beta\,d\theta\,d\beta}{1-\tan^2\theta\,\tan^2\beta} &= \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\cos\theta\,\cos\beta\,d\theta\,d\beta}{\cos^2\theta\,\cos^2\beta-\sin^2\theta\,\sin^2\beta} = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\cos\theta\,\cos\beta\,d\theta\,d\beta}{\cos^2\theta-\sin^2\beta} \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[\log\left(\frac{\cos\theta+\sin\beta}{\cos\theta-\sin\beta}\right)\right]_0^{\frac{\pi}{4}} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \log\left(\frac{\cos\theta+\sin\frac{\pi}{4}}{\cos\theta-\sin\frac{\pi}{4}}\right) d\theta \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{4}} \log\left(\frac{\cos\theta-\cos\frac{\pi}{4}}{\cos\theta+\cos\frac{\pi}{4}}\right) d\theta \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{4}} \log\left(\tan\left(\frac{\pi}{8}+\frac{\theta}{2}\right)\tan\left(\frac{\pi}{8}-\frac{\theta}{2}\right)\right) d\theta \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{4}} \log\left(\tan\left(\frac{\pi}{8}+\frac{\theta}{2}\right)\right) - \frac{1}{2}\int_0^{\frac{\pi}{4}} \log\left(\tan\left(\frac{\pi}{8}-\frac{\theta}{2}\right)\right) d\theta \\
&= -\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \log\varphi\,d\varphi - \int_0^{\frac{\pi}{8}} \log\phi\,d\phi = -\int_0^{\frac{\pi}{4}} \log\left(\tan\omega\right) d\omega = G;
\end{aligned}$$

where the last equality is one of known integral representations of the Catalan's constant. □

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