SOLUTION TO PROBLEM #12120

Problem #12120. Proposed by M. Bataille, France. For positive integers n and k with $n \geq k$, let $a(n,k) = \sum_{j=0}^{k-1} \binom{n}{j} 3^j$.

(a) Evaluate

$$\lim_{n\to\infty} \frac{1}{4^n} \sum_{k=1}^{n} \frac{a(n,k)}{k}.$$

(b) Evaluate

$$\lim_{n\to\infty} n \left(4^n L - \sum_{k=1}^{n} \frac{a(n,k)}{k} \right)$$

where L is the limit is part (a).

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA.
Swap sums: $\sum_{k=1}^{n} \frac{a(n,k)}{k} = \sum_{j=0}^{n} \binom{n}{j} 3^j \sum_{k=j+1}^{n} \frac{1}{k} = \sum_{j=0}^{n} \binom{n}{j} 3^j (H_n - H_j)$ where $H_m = \sum_{k=1}^{m} \frac{1}{k}$.

$$\begin{aligned}
\frac{1}{4^n} \sum_{k=1}^{n} \frac{a(n,k)}{k} &= \frac{1}{4^n} \sum_{j=1}^{n} \binom{n}{j} 3^j \int_0^1 \frac{x^j - x^n}{1-x}\, dx = \frac{1}{4^n} \int_0^1 \frac{1}{1-x} \sum_{j=0}^{n} \binom{n}{j} [3^j x^j - 3^j x^n] dx \\
&= \frac{1}{4^n} \int_0^1 \frac{(1+3x)^n - (4x)^n}{1-x} dx = \int_0^1 \frac{(\frac{1}{4} + \frac{3}{4}x)^n - x^n}{1-x} \\
&= \int_0^1 \frac{(1 - \frac{3}{4}(1-x))^n - 1}{1-x} dx + \int_0^1 \frac{1-x^n}{1-x} dx \\
&= \int_0^1 \frac{(1-\frac{3}{4}t)^n - 1}{t}\, dt + \int_0^1 \frac{1-x^n}{1-x} dx = \int_0^1 \frac{(1-\frac{3}{4}t)^n - 1}{t} dt + \int_0^1 \sum_{k=0}^{n-1} x^{k-1} dx \\
&= \int_0^1 -\frac{3}{4} \sum_{k=0}^{n-1} \left(1 - \frac{3}{4}t\right)^k dt + \sum_{k=0}^{n-1} \frac{1}{k+1} \\
&= \sum_{k=0}^{n-1} \frac{(1-\frac{3}{4})^{k+1} - 1}{k+1} + \sum_{k=0}^{n-1} \frac{1}{k+1} = \sum_{k=0}^{n-1} \frac{(1-\frac{3}{4})^{k+1}}{k+1}.
\end{aligned}$$

We arrive at an answer to part (a):

$$\lim_{n\to\infty} \frac{1}{4^n} \sum_{k=1}^{n} \frac{a(n,k)}{k} = \sum_{k=0}^{\infty} \frac{(1-\frac{3}{4})^{k+1}}{k+1} = -\log\left(\frac{3}{4}\right) = \log\left(\frac{4}{3}\right).$$

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Next, we proceed as follows to resolve part (b) of the problem:

$$\begin{aligned}\lim_{n\to\infty} n4^n\left(L-\frac{1}{4^n}\sum_{k=1}^{n}\frac{a(n,k)}{k}\right) &= \lim_{n\to\infty} n4^n\left(\sum_{k=0}^{\infty}\frac{1}{4^{k+1}(k+1)}-\sum_{k=0}^{n-1}\frac{1}{4^{k+1}(k+1)}\right)\\ &= \lim_{n\to\infty} n4^n\sum_{k=n}^{\infty}\frac{1}{4^{k+1}(k+1)} = \lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{n}{4^k(k+n)}\\ &= \sum_{k=1}^{\infty}\frac{1}{4^k}\left(\lim_{n\to\infty}\frac{n}{k+n}\right) = \sum_{k=1}^{\infty}\frac{1}{4^k} = \frac{1}{3},\end{aligned}$$

where we utilized uniform convergence to pass to the limits inside the infinite sum. □

A generalization

Suppose $a+b=1$ and both a and b are positive real numbers. Then,

$$\begin{aligned}\int_0^1 \frac{(a+bx)^n-x^n}{1-x}dx &= \int_0^1 \frac{(a+bx)^n-1}{1-x}dx+\int_0^1\frac{1-x^n}{1-x}dx\\ &= \int_0^1\frac{(1-b(1-x))^n-1}{1-x}dx+\int_0^1\sum_{k=0}^{n-1}x^{k-1}dx\\ &= \int_0^1\frac{(1-bt)^n-1}{t}dt+\int_0^1\sum_{k=0}^{n-1}x^{k-1}dx\\ &= \int_0^1 -b\sum_{k=0}^{n-1}(1-bt)^k dt+\sum_{k=0}^{n-1}\frac{1}{k+1}\\ &= \sum_{k=0}^{n-1}\frac{(1-b)^{k+1}-1}{k+1}+\sum_{k=0}^{n-1}\frac{1}{k+1}\\ &= \sum_{k=0}^{n-1}\frac{(1-b)^{k+1}}{k+1} \qquad \to \qquad -\log(b), \qquad \text{as } n\to\infty.\end{aligned}$$