# SOLUTION TO PROBLEM #12127

*Problem #12127. Proposed by O. Furdui and A. Sintamarian, Romania.* Calculate

$$\int_0^1 \left(\frac{\mathrm{Li}_2(1)-\mathrm{Li}_2(x)}{1-x}\right)^2 dx$$

where $\mathrm{Li}_2(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^2}$ is the dilogarithm function.

*Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA.* It's known that $\mathrm{Li}_2(x)=\int_x^0 \frac{\log(1-t)}{t}dt$. Therefore,

$$\int_0^1 \left(\frac{\mathrm{Li}_2(1)-\mathrm{Li}_2(x)}{1-x}\right)^2 dx=\int_0^1 \frac{1}{(1-x)^2}\left(\int_1^x \frac{\log(1-t)}{t}dt\right)^2 dx$$

may be integrated by parts. Let $u=\left(\int_1^x \cdots\right)^2$ and $dv=\frac{dx}{(1-x)^2}$ so that

$$\begin{aligned}\int_0^1 \left(\frac{\mathrm{Li}_2(1)-\mathrm{Li}_2(x)}{1-x}\right)^2 dx &= -\mathrm{Li}_2^2(1)+\lim_{x\to 1^-}\frac{(\int_0^1\cdots)^2}{1-x}-2\int_0^1\frac{\log(1-x)}{x(1-x)}\int_1^x\frac{\log(1-t)}{t}dt\,dx\\ &= -\mathrm{Li}_2^2(1)-2\int_0^1\left[\frac{\log(1-x)}{x}+\frac{\log(1-x)}{1-x}\right]\int_1^x\frac{\log(1-t)}{t}dt\,dx\\ &= -\mathrm{Li}_2^2(1)-[\mathrm{Li}_2(1)-\mathrm{Li}_2(x)]^2|_0^1-2\int_0^1\frac{\log(1-x)}{1-x}\int_1^x\frac{\log(1-t)}{t}dt\,dx\\ &= -2\int_0^1\frac{\log(1-x)}{1-x}\int_1^x\frac{\log(1-t)}{t}dt\,dx.\end{aligned}$$

Continue with yet another integration by parts $u=\int_1^x\frac{\log(1-t)}{t}dt$ and $dv=\frac{\log(1-x)}{1-x}dx$ so that

$$\begin{aligned}-2\int_0^1\frac{\log(1-x)}{1-x}\int_1^x\frac{\log(1-t)}{t}dt\,dx &= \lim_{x\to 1^-}\left(\log^2(1-x)\int_1^x\frac{\log(1-t)}{t}dt\right)-\int_0^1\frac{\log^3(1-x)}{x}\,dx\\ &= -\int_0^1\frac{\log^3(1-x)}{x}\,dx\end{aligned}$$

On the other hand, after a change of variables and a repeated integration by parts, we derive

$$\int_0^1\frac{\log^3(1-x)}{x}dx=\int_0^1\frac{\log^3 t}{1-t}dt=\sum_{n\geq 0}\int_0^1 t^n\log^3(1-t)\,dt=-6\sum_{n\geq 0}\frac{1}{(n+1)^4}.$$

We conclude with the evaluation $\int_0^1\left(\frac{\mathrm{Li}_2(1)-\mathrm{Li}_2(x)}{1-x}\right)^2dx=6\zeta(4)=\frac{\pi^4}{15}$. □

**Remark.** Both limits shown above can be verified by L'hopital's rule.

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