## SOLUTION TO PROBLEM #12134

*Problem #12134. Proposed by P. Bracken, USA.* Evaluate the series

$$\sum_{n=1}^{\infty}\left(n\sum_{k=n}^{\infty}\frac{1}{k^2}-1-\frac{1}{2n}\right).$$

*Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA.* Let $H_n=\sum_{k=1}^{n}\frac{1}{k}$, then for $N\geq 1$, we proceed as follows:

$$\begin{aligned}T_N:=\sum_{n=1}^{N}\left(n\sum_{k=n}^{\infty}\frac{1}{k^2}-1-\frac{1}{2n}\right)&=\sum_{k=N+1}^{\infty}\frac{1}{k^2}\sum_{n=1}^{N}n+\sum_{k=1}^{N}\frac{1}{k^2}\sum_{n=1}^{k}n-\sum_{n=1}^{N}\left(1+\frac{1}{2n}\right)\\&=\binom{N+1}{2}\sum_{k=N+1}^{\infty}\frac{1}{k^2}+\sum_{k=1}^{N}\frac{k^2+k}{2k^2}-N-\frac{1}{2}H_N=\binom{N+1}{2}\sum_{k=N+1}^{\infty}\frac{1}{k^2}-\frac{1}{2}N.\end{aligned}$$

Using the asymptotic expansion*

$$\sum_{k=N+1}^{\infty}\frac{1}{k^2}=\frac{1}{N}-\frac{1}{2N^2}+O\left(\frac{1}{N^3}\right),$$

we obtain

$$T_N=\frac{N^2+N}{2}\left[\frac{1}{N}-\frac{1}{2N^2}+O\left(\frac{1}{N^3}\right)\right]-\frac{1}{2}N=\frac{1}{4}+O\left(\frac{1}{N}\right)\qquad\longrightarrow\qquad\frac{1}{4}$$

as $N\to\infty$. So, we conclude that

$$\sum_{n=1}^{\infty}\left(n\sum_{k=n}^{\infty}\frac{1}{k^2}-1-\frac{1}{2n}\right)=\frac{1}{4}.\qquad\square$$

**Justification of (*):** It is easy to show that

$$\int_0^{\infty}dt\frac{t\,e^{-Nt}}{e^t-1}=\sum_{k=N+1}^{\infty}\frac{1}{k^2}.$$

Apply integration by parts to get an expansion in $1/N$:

$$\begin{aligned}\int_0^{\infty}dt\frac{t\,e^{-Nt}}{e^t-1}&=\frac{1}{N}+\frac{1}{N}\int_0^{\infty}dt\,e^{-Nt}\frac{e^t(1-t)-1}{(e^t-1)^2}\\&=\frac{1}{N}-\frac{1}{2N^2}+\frac{1}{N^2}\int_0^{\infty}dt\,e^{-Nt}\frac{e^t\left(e^t(t-2)+t+2\right)}{(e^t-1)^3}\\&=\frac{1}{N}-\frac{1}{2N^2}+O\left(\frac{1}{N^3}\right).\end{aligned}$$