SOLUTION TO PROBLEM #12206

Problem #12206. Proposed by S. Stewart (Australia). Prove

$$\sum_{n=1}^{\infty}\frac{\bar{H}_{2n}}{n^2}=\frac{3}{4}\zeta(3)$$

where $\bar{H}_n$ is the n-th skew-harmonic number $\bar{H}_n=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}$.

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Start by re-writing $\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}=\sum_{k=1}^{n}\frac{(-1)^{2k-1}}{2k}+\sum_{k=1}^{n}\frac{(-1)^{2k-1-1}}{2k-1}=-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}\frac{1}{2k-1}$. Using the usual harmonic numbers $H_n=\sum_{k=1}^{n}\frac{1}{k}$, we get $\bar{H}_{2n}=-\frac{1}{2}H_n+H_{2n}-\frac{1}{2}H_n=H_{2n}-H_n$. On the other hand, $\sum_{n=1}^{\infty}\frac{H_{2n}}{n^2}=2\sum_{n=1}^{\infty}\frac{H_n}{n^2}+2\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}$. Using Euler's identity $\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\zeta(3)$, we gather

$$\sum_{n=1}^{\infty}\frac{\bar{H}_{2n}}{n^2}=\sum_{n=1}^{\infty}\frac{H_{2n}}{n^2}-\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\sum_{n=1}^{\infty}\frac{H_n}{n^2}+2\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}-\sum_{n=1}^{\infty}\frac{H_n}{n^2}=2\zeta(3)+2\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}.$$

From the integral form $H_n=\int_0^1\frac{1-x^n}{1-x}\,dx$, the dilogarithm function $\mathrm{Li}_q(y)=\sum_{n=1}^{\infty}\frac{x^n}{n^q}$, integration by parts and noting $\mathrm{Li}_1(x)=-\log(1-x)$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}&=\int_0^1\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-\sum_{n=1}^{\infty}\frac{(-1)^nx^n}{n^2}\right]\frac{dx}{1-x}=\int_0^1\frac{-\frac{1}{2}\zeta(2)-\mathrm{Li}_2(-x)}{1-x}dx\\&=\left[(\zeta(2)+\mathrm{Li}_2(-x))\ln(1-x)\right]_0^1-\int_0^1\ln(1-x)\,\mathrm{Li}_1(-x)\,dx\\&=\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}\,dx.\end{aligned}$$

To evaluate the last integral observe that

$$\begin{aligned}\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}\,dx&=\frac{1}{4}\int_0^1\frac{\ln^2(1-x^2)}{x}dx-\frac{1}{4}\int_0^1\frac{\ln^2\left(\frac{1-x}{1+x}\right)}{x}dx\\&=\frac{1}{8}\int_0^1\frac{\ln^2(1-t)}{t}dt-\frac{1}{2}\int_0^1\frac{\ln^2 t}{1-t^2}dt=\frac{1}{8}\int_0^1\frac{\ln^2 t}{1-t}dt-\frac{1}{2}\int_0^1\frac{\ln^2 t}{1-t^2}dt\\&=\frac{1}{8}\sum_{n\geq0}\int_0^1t^n\ln^2t\,dt-\frac{1}{2}\sum_{n\geq0}t^{2n}\ln^2t\,dt.\end{aligned}$$

Now, proceed with

$$\int_0^1t^bdt=\frac{1}{b+1}\implies\int_0^1t^b\ln^kt\,dt=\frac{d^k}{db^k}\frac{1}{b+1}=\frac{(-1)^kk!}{(b+1)^{k+1}},$$

$$\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}\,dx=\frac{1}{8}\sum_{n\geq0}\frac{2}{(n+1)^3}-\frac{1}{2}\sum_{n\geq0}\frac{2}{(2n+1)^3}=\frac{2}{3}\zeta(3)-\frac{7}{8}\zeta(3)=-\frac{5}{8}\zeta(3).$$

Combining it all, we conclude that $\sum_{n=1}^{\infty}\frac{\bar{H}_{2n}}{n^2}=2\zeta(3)+2\left(-\frac{5}{8}\right)=\frac{3}{4}\zeta(3)$. □

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