SOLUTION TO PROBLEM #12206

Problem #12206. Proposed by R. Tauraso(Italy). Prove

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=k}^{\infty}\frac{1}{2^n n}=-\frac{13\zeta(3)}{24}.$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Start with $\int_0^1 t^{b-1}dt=\frac{1}{b}$ so that $\int_0^1 t^{b-1}\log t\,dt=\frac{d}{db}\left(\frac{1}{b}\right)=-\frac{1}{b^2}$. Also that $\int_0^{\frac{1}{2}}x^{n-1}dx=\frac{1}{2^n n}$. Thus,

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=k}^{\infty}\frac{1}{2^n n}=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=1}^{\infty}\frac{1}{2^n n}-\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=1}^{k-1}\frac{1}{2^n n}=-\frac{\log(2)\zeta(2)}{2}-\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=1}^{k-1}\frac{1}{2^n n}.$$

We focus on the last double sum:

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=1}^{k-1}\frac{1}{2^n n}&=\int_0^1\sum_{k=1}^{\infty}(-t)^{k-1}\log t\,dt\int_0^{\frac{1}{2}}\sum_{n=1}^{k-1}x^{n-1}dx\\
&=\int_0^1\sum_{k=1}^{\infty}(-t)^{k-1}\log t\,dt\int_0^{\frac{1}{2}}\frac{1-x^{k-1}}{1-x}dx\\
&=\int_0^1\int_0^{\frac{1}{2}}\log t\sum_{k=1}^{\infty}\left(\frac{(-t)^{k-1}-(-tx)^{k-1}}{1-x}\right)dx\,dt\\
&=\int_0^1\left(\int_0^{\frac{1}{2}}\frac{\log t}{(1-x)(1+t)}-\frac{\log t}{(1-x)(1+tx)}\right)dx\,dt\\
&=\log 2\int_0^1\frac{\log t}{1+t}dt-\int_0^1\frac{\log t\cdot\log(2+t)}{1+t}dt\\
&=-\frac{\log(2)\zeta(2)}{2}-\int_0^1\frac{\log t\cdot\log(2+t)}{1+t}dt.
\end{aligned}$$

So far, we have $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\sum_{n=k}^{\infty}\frac{1}{2^n n}=\int_0^1\frac{\log t\cdot\log(2+t)}{1+t}\,dt=\int_1^2\frac{\log(x-1)\cdot\log(x+1)}{x}\,dx.$

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