SOLUTION TO PROBLEM #12249

Problem #12249. Proposed by F. Stanescu (Romania). Prove Prove

$$\sum_{k=\lfloor n/2\rfloor}^{n-1}\sum_{m=1}^{n-k}(-1)^{m-1}\frac{k+m}{k+1}\binom{k+1}{m-1}2^{k-m}=\frac{n}{2}$$

for any positive integer n.

Solution by Tewodros Amdeberhan, Tulane University and Akalu Tefera, Grand Valley State University, MI , USA. Let the inner sum be $F_1(k,m)=(-1)^{m-1}\frac{k+m}{k+1}\binom{k+1}{m-1}2^{k-m}$ and $G_1(k,m)=2(-1)^m\binom{k}{m-2}2^{k-m}$. One can check that $F_1(k,m)=G_1(k,m+1)-G_1(k,m)$. So, we get a telescoping sum

$$\begin{aligned}\sum_{m=1}^{n-k}F_1(k,m)&=\sum_{m=1}^{n-k}[G_1(k,m+1)-G_1(k,m)]=G_1(k,n-k+1)-G_1(k,1)\\&=G_1(k,n-k+1)=2(-1)^{n-k+1}\binom{k}{n-k-1}2^{2k-n-1}.\end{aligned}$$

It remain to show that $\sum_{k=\lfloor n/2\rfloor}^{n-1}2(-1)^{n-k+1}\binom{k}{n-k-1}2^{2k-n-1}=\frac{n}{2}$. Denote

$$F_2(n,k)=\frac{(-1)^{n-k+1}}{n}\binom{k}{n-k-1}2^{2k-n+1}\qquad\text{and}\qquad G_2(n,k)=F_2(n,k)\frac{(2k-n)(2k-n+1)}{2(n-k)(n+1)}.$$

Once more, its can be verified that $F_2(n+1,k)-F_2(n,k)=G_2(n,k+1)-G_2(n,k)$. Now, sum both sides over all integers k. Then, the right-hand sides adds up to zero and hence the resulting equation becomes $\sum_{k=\lfloor (n+1)/2\rfloor}^{n}F_2(n+1,k)-\sum_{k=\lfloor n/2\rfloor}^{n}F_2(n,k)=0$. That means the sequence $f_n:=\sum_{k=\lfloor n/2\rfloor}^{n}F_2(n,k)$ is constant. Since $f_1=0$, we conclude that $f_n\equiv 1$. The claim follows. $\square$