SOLUTION TO PROBLEM #12256

Problem #12256. Proposed by P. Bracken (USA). Prove

$$\int_0^1 \frac{\log(1-x)\,\log(1+x)}{x}\,dx = -\frac{5}{8}\zeta(3).$$

Solution by Tewodros Amdeberhanl, Tulane University, and Akalu Tefera, Grand Valley State University, MI, USA. Use $4AB = (A+B)^2 - (A-B)^2$ with $A = \log(1-x)$ and $B = \log(1+x)$ to obtain

$$\int_0^1 \frac{\log(1+x)\,\log(1-x)}{x}\,dx = \frac{1}{4}\int_0^1 \frac{\log^2(1-x^2)}{x}\,dx - \frac{1}{4}\int_0^1 \frac{\log^2\left(\frac{1-x}{1+x}\right)}{x}\,dx.$$

Substitute $w = x^2$ and $u = \frac{1-x}{1+x}$ in the first and second integrals, respectively, so that

$$\begin{aligned}\int_0^1 \frac{\log(1+x)\,\log(1-x)}{x}\,dx &= \frac{1}{8}\int_0^1 \frac{\log^2(1-w)}{w}\,dw - \frac{1}{2}\int_0^1 \frac{\log^2 u}{1-u^2}\,du \\ &= \frac{1}{8}\int_0^1 \frac{\log^2 u}{1-u}\,du - \frac{1}{2}\int_0^1 \frac{\log^2 u}{1-u^2}\,du \\ &= \frac{1}{8}\sum_{k=0}^{\infty}\int_0^1 u^k \log^2 u\,du - \frac{1}{2}\sum_{k=0}^{\infty}\int_0^1 t^{2k}\log^2 u\,du.\end{aligned}$$

Since $\int_0^1 u^t du = \frac{1}{t+1}$, we get $\int_0^1 u^t \log^2 u\,du = \frac{d^2}{dt^2}\left(\frac{1}{t+1}\right) = \frac{2}{(t+1)^3}$. It follows that

$$\begin{aligned}\int_0^1 \frac{\log(1-x)\,\log(1+x)}{x}\,dx &= \frac{1}{8}\sum_{k=0}^{\infty}\frac{2}{(k+1)^3} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{2}{(2n+1)^3} \\ &= \frac{2}{8}\zeta(3) - \frac{7}{8}\zeta(3) \\ &= -\frac{5}{8}\zeta(3).\end{aligned}$$

The proof is complete. □

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