SOLUTION TO PROBLEM #12260

Problem #12260. Proposed by S. M. Stewart (Australia). Prove

$$\int_0^\infty \frac{\sin^2 x - x\sin x}{x^3}\,dx = \frac{1}{2} - \log 2.$$

Solution by Tewodros Amdeberhan, Tulane University, and Akalu Tefera, Grand Valley State Univeristy, MI, USA. Recall this property of the Laplace Transform

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty (\mathcal{L}f)(s)(\mathcal{L}^{-1}g)(s)ds$$

where we take $f(x) = \sin^2 x - x\sin x = \frac{1}{2} - \frac{1}{2}\cos(2x) - x\sin x$ and $g(x) = \frac{1}{x^3}$. This leads to

$$\begin{aligned}
\int_0^\infty \frac{\sin^2 x - x\sin x}{x^3}\,dx &= \int_0^\infty \mathcal{L}\left(\frac{1}{2} - \frac{1}{2}\cos(2x) - x\sin x\right)\cdot \mathcal{L}^{-1}\left(\frac{1}{x^3}\right)\,ds \\
&= \int_0^\infty \left(\frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} - \frac{2s}{(s^2+1)^2}\right)\frac{s^2}{2}\,ds \\
&= \int_0^\infty \frac{s}{(s^2+1)^2}ds + \int_0^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+1}\right)ds \\
&= \left[-\frac{1}{2(s^2+1)}\right]_0^\infty + \left[\frac{\log(s^2+4)}{2} - \frac{\log(s^2+1)}{2}\right]_0^\infty \\
&= \frac{1}{2} - \log 2;
\end{aligned}$$

where we employed partial fraction decomposition to get to the 3^{rd} line. The proof is complete. □

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