SOLUTION TO PROBLEM #12274

Problem #12274. Proposed by R. Tauraso (Italy). Evaluate

$$I := \int_0^1 \frac{\arctan x}{1+x^2} \log^2\left(\frac{2x}{1-x^2}\right) dx.$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Substitute $x = \frac{1-y}{1+y}$ and the identity $\arctan \frac{1-y}{1-y} = \frac{\pi}{4} - \arctan y$ to convert the given integral to

$$I = \int_0^1 \frac{\arctan\left(\frac{1-y}{1+y}\right)}{1+y^2} \log^2\left(\frac{1-y^2}{2y}\right) dy = \frac{\pi}{4}\int_0^1 \frac{\log^2\left(\frac{2y}{1-y^2}\right)}{1+y^2} dy - \int_0^1 \frac{\arctan y}{1+y^2} \log^2\left(\frac{2y}{1-y^2}\right) dy.$$

The integral becomes $I = \frac{\pi}{8}\int_0^1 \frac{\log^2\left(\frac{2y}{1-y^2}\right)}{1+y^2} dy$. Substituting $y = \tan t$ and using identities lead to

$$I = \frac{\pi}{8}\int_0^{\frac{\pi}{4}} \log^2\left(\frac{2\tan t}{1-\tan^2 t}\right) dt = \frac{\pi}{8}\int_0^{\frac{\pi}{4}} \log^2\left(\frac{2\cos t \sin t}{\cos^2 t - \sin^2 t}\right) dt = \frac{\pi}{16}\int_0^{\frac{\pi}{2}} \log^2(\tan t)\, dt.$$

Again, letting $z = \tan t$ transforms the integral to $I = \frac{\pi}{16}\int_0^\infty \frac{\log^2 z}{1+z^2}\, dz$. At this point, consider the *Mellin transform* $F(s) = \int_0^\infty \frac{z^{s-1}}{1+z^2}\, dz = \frac{\pi}{2\sin(\frac{\pi s}{2})}$. Differentiate this twice to get

$$F''(s) = \frac{\pi^3 \cos^2(\frac{\pi s}{2})}{4\sin^3(\frac{\pi s}{2})} + \frac{\pi^3}{8\sin(\frac{\pi s}{2})} \qquad \Longrightarrow \qquad F''(1) = \int_0^\infty \frac{\log^2 z}{1+z^2}\, dz = \frac{\pi^3}{8}.$$

We conclude that $I = \frac{\pi^4}{128}$ completes the solution. $\square$