SOLUTION TO PROBLEM #12276

Problem #12276. Proposed by J. Santmyer, Las Cruces, NM . Prove

$$\sum_{n=2}^{\infty}\frac{1}{n+1}\sum_{i=1}^{\lfloor n/2\rfloor}\frac{1}{2^{i-1}(i-1)!(n-2i)!}=1.$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Start with the exponential generating function for the number of involutions I_m, given in the form $e^{x+\frac{1}{2}x^2}=\sum_{m=0}^{\infty}\frac{I_m}{m!}x^m$. It is also well-known $I_m=\sum_{k=0}^{\lfloor m/2\rfloor}\frac{m!}{2^k k!(m-2k)!}$. Now, proceed as follows:

$$\begin{aligned}x^2e^{x+\frac{1}{2}x^2}&=\sum_{m=0}^{\infty}\sum_{k=0}^{\lfloor m/2\rfloor}\frac{x^{m+2}}{2^k k!(m-2k)!}=\sum_{n=2}^{\infty}\sum_{k=0}^{\lfloor (n-2)/2\rfloor}\frac{x^n}{2^k k!(n-2-2k)!}\\&=\sum_{n=2}^{\infty}x^n\sum_{i=1}^{\lfloor n/2\rfloor}\frac{1}{2^{i-1}(i-1)!(n-2i)!}.\end{aligned}$$

Integrate both sides over the range $0\le x\le 1$ to find $\int_0^1 x^2e^{x+\frac{1}{2}x^2}dx=(x-1)e^{x+\frac{1}{2}x^2}|_0^1=1$ while

$$\int_0^1\sum_{n=2}^{\infty}x^n\sum_{i=1}^{\lfloor n/2\rfloor}\frac{1}{2^{i-1}(i-1)!(n-2i)!}dx=\sum_{n=2}^{\infty}\frac{1}{n+1}\sum_{i=1}^{\lfloor n/2\rfloor}\frac{1}{2^{i-1}(i-1)!(n-2i)!}.$$

The proof is complete. □