SOLUTION TO PROBLEM #12279

Problem #12279. Proposed by B. Isaacson (USA). Let $S(n,k)$ denote the number of partitions of a set with n elements into k nonempty blocks. (These are the *Stirling numbers of the second kind.*) Let j and n be positive integers of opposite parity with $j<n$. Prove

$$\sum_{k=j}^{n}\frac{(-1)^k(k-1)!\binom{k}{j}\,S(n,k)}{2^k}=0.$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. The convention $\binom{a}{b}=0$ if $a<b$ and multiplying by $(-2)^n$ (turning the sum into an integer), denote

$$f(n,j):=\sum_{k=0}^{n}(-2)^{n-k}(k-1)!\binom{k}{j}\,S(n,k).$$

We wish to show $f(n,j)=(j-1)f(n-1,j-1)-(j+1)f(n-1,j+1)$. Using the familiar recurrence $S(n,k)=kS(n-1,k)+S(n-1,k-1)$, this amounts to

$$\sum_{k=0}^{n}\frac{k!\binom{k}{j}S(n-1,k)}{(-2)^k}+\sum_{k=0}^{n}\frac{(k-1)!\binom{k}{j}S(n-1,k-1)}{(-2)^k}=\sum_{k=0}^{n}\frac{(k-1)!\binom{k-2}{j-2}\,S(n-1,k-1)}{(-2)^k}-\sum_{k=0}^{n-1}\frac{k!\binom{k-1}{j}\,S(n-1,k)}{(-2)^k+1}.$$

The next step is to reindex each summand to reflect a term $S(n-1,k-1)$ and collect every sum:

$$\sum_{k=0}^{n}\frac{(k-1)!S(n-1,k-1)}{(-2)^k}\left[(-2)\binom{k-1}{j}+\binom{k-2}{j}-\binom{k-2}{j-2}+\binom{k}{j}\right]=0.$$

However, the entity inside the square brackets resolves to 0, so the assertion holds true immediately. Now, consider $T_n:=(-1)^{n-1}\frac{d^{n-1}\tanh x}{dx^{n-1}}$. The special case $T_2=-\frac{d\tanh x}{dx}=-\text{sech}^2x=-1+\tanh^2 x$ shows that $T_n=\sum_j b(n,j)\tanh^j x$ so that $b(n,j)$ comes from that of T_{n-1} due to the derivatives of

$$\frac{d\,\left[b(n-1,j-1)\tanh^{j-1}x\right]}{dx}=(j-1)b(n-1,j-1)(-1+\tanh x)\tanh^{j-2}x,$$
$$\frac{d\,\left[b(n-1,j+1)\tanh^{j+1}x\right]}{dx}=(j+1)b(n-1,j+1)(-1+\tanh x)\tanh^{j}x$$

satisfying the recurrence $b(n,j)=(j-1)b(n-1,j-1)-(j+1)b(n-1,j+1)$, which agrees with that of $f(n,j)$. The boundary conditions can be easily checked to agree, too. Furthermore, there is the apparent $b(n,j)=0$ if n and j are of opposite parity which is exactly the required assertion. $\square$

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