SOLUTION TO PROBLEM #12281

Problem #12281. Proposed by P. Perfetti (Italy). Evaluate

$$I := \int_0^\infty \left(\frac{\cosh x}{\sinh^2 x} - \frac{1}{x^2}\right) \ln^2 x \, dx.$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Integrate by parts: let $u = \ln^2 x, dv = \left(\frac{\cosh x}{\sinh^2 x} - \frac{1}{x^2}\right) dx$. Thus $du = \frac{2\ln x}{x}\, dx, v = \frac{1}{x} - \frac{1}{\sinh x}$ and

$$I = \left[\left(\frac{1}{x} - \frac{1}{\sinh x}\right) \ln^2 x\right]_0^\infty - 2\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right) \frac{\ln x}{x}\, dx = 2\int_0^\infty \left(\frac{1}{\sinh x} - \frac{1}{x}\right) \frac{\ln x}{x}\, dx.$$

Let B_{2n} denote *Bernoulli* numbers. Use the Laurent series $\frac{1}{\sinh x} = \sum_{n=0}^\infty \frac{2(1-2^{2n-1})B_{2n}x^{2n-1}}{(2n)!}$ and consider a new (but related) integral

$$\begin{aligned} J(s) :&= \int_0^\infty \left(\frac{1}{\sinh x} - \frac{1}{x}\right) \frac{x^s}{x} dx = \int_0^\infty x^s \sum_{\boldsymbol{n=1}}^\infty \frac{2(1-2^{2n-1})B_{2n}x^{2n-2}}{(2n)!}\, dx \\ &= \int_0^\infty x^s \sum_{k=0}^\infty \frac{2(1-2^{2k+1})B_{2k+2}x^{2k}}{(2k+2)!}\, dx = \int_0^\infty u^{\boldsymbol{\frac12(s+1)}-1} \sum_{k=0}^\infty \frac{(1-2^{2k+1})B_{2k+2}u^k}{(2k+2)!}\, du \\ &= \int_0^\infty u^{\boldsymbol{\frac12(s+1)}-1} \sum_{k=0}^\infty \frac{2(1-2^{2k+1})\,\zeta(2k+2)}{(2\pi)^{2k+2}} (-u)^k\, du; \end{aligned}$$

where the connection between B_{2n} and the *Riemann zeta* function $\zeta(z)$ is utilized. It's time to invoke *Ramanujan's Master Theorem* $\int_0^\infty u^{t-1} \sum_{k=0}^\infty \varphi(k)\,(-u)^k\, du = \frac{\pi}{\sin(t\pi)}\,\varphi(-t)$ with $t = \boldsymbol{\frac12(s+1)}$ and $\varphi(k) = \frac{2(1-2^{2k+1})\,\zeta(2k+2)}{(2\pi)^{2k+2}}$. Therefore, we have $J(s) = \frac{\pi}{\sin(\frac12(s+1)\pi)} \cdot \frac{2(1-2^{-s})\zeta(-s+1)}{(2\pi)^{-s+1}}$ which we prefer to express in terms of *Dirichlet's eta* function $\eta(1-s) = (1-2^s)\zeta(1-s)$ (avoiding singularities)

$$\begin{aligned} I = 2\frac{dJ}{ds}(0) &= \frac{d}{ds}\left[\frac{2(1-2^{-s})(2\pi)^s\zeta(1-s)}{\cos(\frac12 s\pi)}\right]_{s=0} = -\frac{d}{ds}\left[\frac{2^{1-s}(2\pi)^s\eta(1-s)}{\cos(\frac12 s\pi)}\right]_{s=0} \\ &= -\left[\frac{2^{1-s}\ln(\frac12)(2\pi)^s\eta(1-s)}{\cos(\frac12 s\pi)} + \frac{2^{1-s}(2\pi)^s\ln(2\pi)\eta(1-s)}{\cos(\frac12 s\pi)}\right]_{s=0} \\ &\qquad + \left[\frac{2^{1-s}(2\pi)^s\eta'(1-s)}{\cos(\frac12 s\pi)} + \frac{2^{1-s}\pi\sin(\frac12 s\pi)(2\pi)^s\eta(1-s)}{2\cos^2(\frac12 s\pi)}\right]_{s=0} \\ &= [2\gamma - \ln(2) - 2\ln(\pi)]\ln(2). \end{aligned}$$

Here γ is *Euler's constant.* The solution is now complete. □

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