SOLUTION TO PROBLEM #12286

Problem #12286. Proposed by I. Gessel (USA). Let p be a prime number, and let m be a positive integer not divisible by p. Show that the coefficients of $(1+x+\cdots+x^{m-1})^{p-1}$ that are not divisible by p are alternately 1 and -1 modulo p. For example, $(1+x+x^2+x^3)^4 \equiv 1-x+x^4-x^6+x^8-x^{11}+x^{12}$ mod 5.

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. The case $p=2$ is trivial, so assume p is odd. Notice $1+x^m+\cdots+x^{(p-1)m} \equiv (1-x^m)^{p-1}$ mod p. Then, modulo p, we have

$$\begin{aligned}(1+x+\cdots+x^{m-1})^{p-1} &= \left(\frac{1-x^m}{1-x}\right)^{p-1} = (1-x)\,\frac{(1-x^m)^{p-1}}{(1-x)^p} \equiv (1-x)\,\frac{1+x^m+\cdots+x^{(p-1)m}}{1-x^p} \\ &= (1-x)\,(1+x^m+\cdots+x^{(p-1)m})\,(1+x^p+x^{2p}+\cdots) \\ &= (1-x)\sum_{k,j} x^{mk+pj}.\end{aligned}$$

Observe that each term x^{mk+pj} appears only once since $\gcd(m,p)=1$ and $0 \leq k < p$. Thus, the non-vanishing summands in the above sum $\sum_{k,j}$ all have coefficients equal to 1. Upon multiplying by $(1-x)$, the assertion follows immediately. □

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