SOLUTION TO PROBLEM #12287

Problem #12287. Proposed by O. Furdui and A. Sintamarian (Romania). Prove

$$\sum_{n=1}^{\infty}\left(n\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^2-\frac{1}{n}\right)=\frac{3}{2}-\frac{1}{2}\zeta(2)+\frac{3}{2}\zeta(3),$$

where $\zeta(s)$ is the Riemann zeta function, defined by $\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}$.
Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Let S_N be the partial sums of the given series. We perform successive steps: rewrite the summands and change the order of summation.

$$\begin{aligned} S_N &= \sum_{n=1}^{N}\sum_{k=n}^{\infty} n\left(\zeta(2)-\sum_{j=1}^{n-1}\frac{1}{j^2}\right)\frac{1}{k^2}-H_N \\ &= \left(\zeta(2)-\sum_{k=1}^{N}\frac{1}{k^2}\right)\sum_{n=1}^{N} n\left(\zeta(2)-\sum_{j=1}^{n-1}\frac{1}{j^2}\right)+\sum_{k=1}^{N}\frac{1}{k^2}\sum_{n=1}^{k} n\left(\zeta(2)-\sum_{j=1}^{n-1}\frac{1}{j^2}\right)-H_N; \end{aligned}$$

where $H_N=\sum_{j=1}^{N}\frac{1}{j}$. Denote $H_N^{(2)}=\sum_{j=1}^{N}\frac{1}{j^2}$. Let's resolve the following particular term:

$$\begin{aligned} \sum_{n=1}^{\ell} n\left(\zeta(2)-\sum_{j=1}^{n-1}\frac{1}{j^2}\right) &= \binom{\ell+1}{2}\zeta(2)-\sum_{n=1}^{\ell} n\sum_{j=1}^{n-1}\frac{1}{j^2}=\binom{\ell+1}{2}\zeta(2)-\sum_{j=1}^{\ell-1}\frac{1}{j^2}\sum_{n=j+1}^{\ell} n \\ &= \binom{\ell+1}{2}\zeta(2)-\binom{\ell+1}{2}H_\ell^{(2)}+\frac{\ell}{2}+\frac{H_\ell}{2}. \end{aligned}$$

We apply this evaluation twice, once with $\ell=N$ and another with $\ell=k$. After some routine (though tedious) simplifications, we are lead to

$$\begin{aligned} S_N = N\left(\zeta(2)-H_N^{(2)}\right)+H_N(\zeta(2)-H_N^{(2)})+\frac{1}{2}N^2(\zeta(2)-H_N^{(2)})^2+\frac{1}{2}N(\zeta(2)-H_N^{(2)})^2 \\ +\frac{1}{2}\sum_{k=1}^{N}\frac{H_k}{k^2}+\frac{1}{2}\sum_{k=1}^{N}\frac{H_{k-1}}{k^2}-\frac{1}{2}H_N^{(2)}. \end{aligned}$$

At this juncture, use *Euler's identity* $\sum_{k=1}^{\infty}\frac{H_k}{k^2}=2\zeta(3)$ and its relative $\sum_{k=1}^{\infty}\frac{H_{k-1}}{k^2}=\zeta(3)$, also invoke *Stolz-Cesaro Theorem* to the effect that

$$\lim_{N\to\infty} N(\zeta(2)-H_N^{(2)})=\lim_{N\to\infty}\frac{-\frac{1}{(N+1)^2}}{\frac{1}{N+1}-\frac{1}{N}}=1.$$

Finally, $\lim_{N\to\infty} S_N=\frac{3}{2}-\frac{1}{2}\zeta(2)+\frac{3}{2}\zeta(3)$. The proof is now complete. □

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