SOLUTION TO PROBLEM #12288

Problem #12288. Proposed by S. Stewart (Australia). Prove

$$\int_0^\infty \left(1 - x^2\sin^2\left(\frac{1}{x}\right)\right)^2 dx = \frac{\pi}{5}.$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA.
Denote $I := \int_0^\infty \left(1 - x^2\sin^2\left(\frac{1}{x}\right)\right)^2 dx$. Substitute $x = u^{-1/2}$ so that $dx = -\frac{1}{2}u^{-3/2}du$. When combined with the double-angle formulas $2\sin^2 w = 1 - \cos(2w)$ and $2\cos^2 w = 1 + \cos(2w)$, we find

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty \left(1 - \frac{\sin^2(\sqrt{u})}{u}\right)^2 u^{-3/2}du \\ &= \frac{1}{2}\int_0^\infty \left(1 - \frac{1}{u} + \frac{3}{8u^2} + \frac{(2u-1)\cos(2\sqrt{u})}{2u^2} + \frac{\cos(4\sqrt{u})}{8u^2}\right) u^{-3/2}du. \end{aligned}$$

Now, apply the Taylor series $\cos w = \sum_{k=0}^\infty (-1)^k \frac{w^{2k}}{(2k)!}$ and regroup the terms to arrive at

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty \left(\sum_{k=2}^\infty \frac{(-1)^k 2^{2k} u^{k-1}}{(2k)!} - \sum_{k=3}^\infty \frac{(-1)^k 2^{2k-1} u^{k-2}}{(2k)!} + \sum_{k=3}^\infty \frac{(-1)^k 2^{4k-3} u^{k-2}}{(2k)!}\right) u^{-3/2}du \\ &= \int_0^\infty \left(\sum_{n=0}^\infty \frac{(-1)^n 2^{2n+3} u^n}{(2n+4)!} + \sum_{n=0}^\infty \frac{(-1)^n 2^{2n+4} u^n}{(2n+6)!} - \sum_{n=0}^\infty \frac{(-1)^n 2^{4n+8} u^n}{(2n+6)!}\right) u^{\mathbf{1/2}-1}du \\ &= \int_0^\infty u^{\mathbf{1/2}-1} \sum_{n=0}^\infty \left(\frac{2^{2n+3}}{\Gamma(2n+5)} + \frac{2^{2n+4}}{\Gamma(2n+7)} - \frac{2^{4n+8}}{\Gamma(2n+7)}\right)(-u)^n du. \end{aligned}$$

Next, implement *Ramanujan's Master Theorem* (appropriate growth conditions are easily checked)

$$\int_0^\infty u^{s-1}\sum_{n=0}^\infty \varphi(n)\cdot(-u)^n du = \frac{\pi}{\sin(s\pi)}\,\varphi(-s)$$

with $\varphi(n) = \frac{2^{2n+3}}{\Gamma(2n+5)} + \frac{2^{2n+4}}{\Gamma(2n+7)} - \frac{2^{4n+8}}{\Gamma(2n+7)}$ and $s = \mathbf{1/2}$. The outcome is this:

$$I = \frac{\pi}{\sin(\frac{\mathbf{1}}{\mathbf{2}}\pi)}\,\varphi(-\mathbf{1/2}) = \pi\left(\frac{2^{-\mathbf{1}+3}}{\Gamma(-\mathbf{1}+5)} + \frac{2^{-\mathbf{1}+4}}{\Gamma(-\mathbf{1}+7)} - \frac{2^{-\mathbf{2}+8}}{\Gamma(-\mathbf{1}+7)}\right) = \frac{\pi}{5}.$$

The proof is complete. □

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