SOLUTION TO PROBLEM #12289

Problem #12289. Proposed by G. E. Andrews (USA) and M. Merca (Romania). Prove

$$\sum_{n=0}^{\infty} 2\cos\left(\frac{(2n+1)\pi}{3}\right) q^{\binom{n+1}{2}} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{6n-1})(1-q^{6n-5})$$

when $|q|<1$.

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Schur's partition identity shows $\prod_{n=1}^{\infty}(1-q^n)(1-q^{6n-1})(1-q^{6n-5}) = \prod_{n=1}^{\infty}\frac{1-q^n}{(1+q^{3n-1})(1+q^{3n-2})}$. On the other hand, $\sum_{n=0}^{\infty} 2\cos\left(\frac{(2n+1)\pi}{3}\right) q^{\binom{n+1}{2}} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} - 3q\sum_{n=0}^{\infty} q^{9\binom{n+1}{2}}$. The rest follows from Bruce Berndt, *Ramanujan's Notebooks*, Part III, page 349, Entry 2 (ii) restated as

$$\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} - 3q\sum_{n=0}^{\infty} q^{9\binom{n+1}{2}} = \prod_{n=1}^{\infty}\frac{(1-q^n)(1-q^{6n})}{(1+q^n)(1-q^{3n})} = \prod_{n=1}^{\infty}\frac{1-q^n}{(1+q^{3n-1})(1+q^{3n-2})}. \qquad \square$$

REMARK. The latter identity can be reproved in a different way than what appeared in the reference indicated above. Let $\eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$. After some reorganization the last identity can be pictured in the form

$$\frac{\eta^4(q^4)\eta^2(q^6)}{\eta^2(q^2)} - 3\frac{\eta^2(q^4)\eta^2(q^6)\eta^2(q^{36})}{\eta(q^2)\eta(q^{18})} = \eta(q^2)\eta(q^4)\eta(q^6)\eta(q^{12}).$$

Each of the three terms in the above equation are in the space of weight 2 cusp forms for the group $\Gamma_0(72)$. This space has dimension 5, and if $f = \sum_{n=1}^{\infty} a_n q^n$ is a function in this space with $a_1 = a_2 = \cdots = a_7 = 0$, it follows that $f = 0$. Hence, it suffices to verify that the coefficients of q^1 through q^7 are the same for the two sides of the identity at hand.