SOLUTION TO PROBLEM #12293

Problem #12293. Proposed by H. Ohtsuka (Japan) and R. Tauraso (Italy). For any integer $n \geq 1$, and any real number $r > 0$, prove

$$\sum_{k=0}^{n}(-1)^k \left(\sum_{j=0}^{k} r^j \binom{n}{j}\right)\left(\sum_{j=0}^{k}(-r)^j \binom{n}{j}\right) = \left(\frac{(r+1)^n + (r-1)^n}{2}\right)^2.$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. From Binomial Theorem, $\frac{(r+1)^n+(r-1)^n}{2} = \sum_{j=0}^{n} \frac{r^{n-j}\binom{n}{j}+r^{n-j}(-1)^j\binom{n}{j}}{2} = \sum_{j=0}^{\lfloor n/2 \rfloor} r^{n-2j}\binom{n}{2j}$. Using the identity $ab = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2$ with $a = \sum_{j=0}^{k} r^j\binom{n}{j}$ and $b = \sum_{j=0}^{k}(-r)^j\binom{n}{j}$, we proceed as:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k a\cdot b &= \sum_{k=0}^{n}(-1)^k \left(\sum_{j=0}^{k} \frac{r^j\binom{n}{j} + (-r)^j\binom{n}{j}}{2}\right)^2 - \sum_{k=0}^{n}(-1)^k \left(\sum_{j=0}^{k} \frac{r^j\binom{n}{j} - (-r)^j\binom{n}{j}}{2}\right)^2 \\ &= \sum_{k=0}^{n}(-1)^k \left(\sum_{j=0}^{\lfloor k/2 \rfloor} r^{2j}\binom{n}{2j}\right)^2 - \sum_{k=1}^{n}(-1)^k \left(\sum_{j=0}^{\lfloor (k-1)/2 \rfloor} r^{2j+1}\binom{n}{2j+1}\right)^2,\end{aligned}$$

Successive terms in $\sum_{k=0}^{n}(-1)^k \left(\sum_{j=0}^{\lfloor k/2 \rfloor}\right)^2$ as well as in $\sum_{k=1}^{n}(-1)^k \left(\sum_{j=0}^{\lfloor (k-1)/2 \rfloor}\right)^2$ cancel pair-wise. Assume $n \to 2n$ is even. In this case, second the double sum vanishes while the first double sum retains one summand (for $k = 2n$), i.e. $\left(\sum_{j=0}^{n} r^{2j}\binom{2n}{2j}\right)^2$. This agrees with $\left(\frac{(r+1)^{2n}+(r-1)^{2n}}{2}\right)^2 = \left(\sum_{j=0}^{n} r^{2n-2j}\binom{2n}{2j}\right)^2$. The argument is similar if $n \to 2n+1$ is odd. This time the first double sum vanishes while the second double sum maintains a single summand (for $k = 2n+1$), i.e. $\left(\sum_{j=0}^{n} r^{2j+1}\binom{2n+1}{2j+1}\right)^2$. Again, this matches $\left(\frac{(r+1)^{2n+1}+(r-1)^{2n+1}}{2}\right)^2 = \left(\sum_{j=0}^{n} r^{2n+1-2j}\binom{2n+1}{2j}\right)^2$. The proof is now complete. $\square$