SOLUTION TO PROBLEM #12297

Problem #12297. Proposed by N. Bhandari (Nepal). Prove

$$\int_0^{\frac{\pi}{2}} \left(\frac{\operatorname{arcsinh}(\sin x)}{\sin x}\right)^2 dx = \frac{\pi}{2}\left(\frac{\pi}{2} - \log 2\right).$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. These known facts $\frac{\operatorname{arcsinh} x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\binom{2n}{n}\frac{x^{2n}}{2n+1}$, $\int_0^{\frac{\pi}{2}} (\sin x)^{2n}\,dx = \frac{\pi}{2}\binom{2n}{n}\frac{1}{4^n}$ and Cauchy's product formula lead to

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \left(\frac{\operatorname{arcsinh}(\sin x)}{\sin x}\right)^2 dx &= \sum_{k,j=0}^{\infty} \frac{(-1)^{k+j}}{4^{k+j}(2k+1)(2j+1)}\binom{2k}{k}\binom{2j}{j}\int_0^1 (\sin x)^{2k+2j}\,dx \\
&= \frac{\pi}{2}\sum_{k,j=0}^{\infty} \frac{(-1)^{k+j}}{4^{k+j}(2k+1)(2j+1)}\binom{2k}{k}\binom{2j}{j}\frac{1}{4^{k+j}}\binom{2k+2j}{k+j} \\
&= \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(2k+1)(2n-2k+1)} \\
&= \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}\binom{2n}{n}\frac{1}{n+1}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{2k+1}.
\end{aligned}$$

We claim that $\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{2k+1} = \frac{16^n}{(2n+1)\binom{2n}{n}}$. We show this via the Wilf-Zeilberger method. Let $F(n,k) = \binom{2k}{k}\binom{2n-2k}{n-k}\binom{2n}{n}\frac{2n+1}{16^n}$ and $G(n,k) = -\binom{2k}{k}\binom{2n-2k+1}{n-k}\binom{2n+1}{n}\frac{k}{4(n+1)16^n}$. Now, check $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$ and sum both sides over all integers k. It turns out that $\sum_{k=0}^{n+1} F(n+1,k) = \sum_{k=0}^{n} F(n,k)$. For $n=0$, this common sum equals 1. The claim follows. Therefore, we have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \left(\frac{\operatorname{arcsinh}(\sin x)}{\sin x}\right)^2 dx &= \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}\binom{2n}{n}\frac{1}{n+1}\frac{16^n}{(2n+1)\binom{2n}{n}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(2n+1)} \\
&= \frac{\pi}{2}\left(2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} - \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\right) = \frac{\pi}{2}\left(\frac{\pi}{2} - \log 2\right).
\end{aligned}$$

The proof is complete. □

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