SOLUTION TO PROBLEM #12298

Problem #12298. Proposed by G. Stoica (Canada). Let n be a positive integer, S_n be the group of all permutations of $\{1,2,\ldots,n\}$, and z be a primitive complex n-th root of unity. Prove

$$\sum_{\sigma\in S_n}\prod_{j=1}^{n}\left(1-x_j z^{\sigma(j)}\right)=n!\left(1-\prod_{k=1}^{n}x_k\right)$$

for any $x_1,x_2,\ldots,x_n\in\mathbb{C}$.

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{e}_k(\boldsymbol{x})=\sum_{j_1<\cdots<j_k}x_1^{j_1}\cdots x_k^{j_k}$ be the k-th elementary symmetric polynomial with $\boldsymbol{e}_0(\boldsymbol{x})=1$. Denote $\xi_j=z^j$ so that $\boldsymbol{X}^n-1=(\boldsymbol{X}-\xi_1)\cdots(\boldsymbol{X}-\xi_n)=\sum_{k=0}^{n}(-1)^k\boldsymbol{e}_k(\boldsymbol{\xi})\boldsymbol{X}^{n-k}$. Then, $\boldsymbol{e}_k(\boldsymbol{\xi})=0$ for $0<k<n$ and $\boldsymbol{e}_0(\boldsymbol{\xi})=1,\boldsymbol{e}_n(\boldsymbol{\xi})=(-1)^{n-1}$. Here $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_n)$.

By construction, the given sum $\sum_\sigma\prod_j(1-x_j\xi_{\sigma(j)})=\sum_\sigma\prod_j(1-x_{\sigma^{-1}(j)}\xi_j)$ is symmetric in both sets of variables ξ and $\boldsymbol{x}$. It is now easy to rewrite the left-hand side of the claim as

$$\sum_{\sigma\in S_n}\prod_{j=1}^{n}(1-x_j\xi_{\sigma(j)})=\sum_{k=0}^{n}(-1)^k\frac{n!}{\binom{n}{k}}\boldsymbol{e}_k(\boldsymbol{x})\boldsymbol{e}_k(\boldsymbol{\xi})=n!-n!\,\boldsymbol{e}_n(\boldsymbol{x})=n!\left(1-\prod_{k=1}^{n}x_k\right).$$

The proof is now complete. □

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