SOLUTION TO PROBLEM #12302

Problem #12302. Proposed by M. Omarjee (France). Let n be a positive integer, and let A_{2n} be the $2n$-by-$2n$ skew-symmetric matrix with (i,j)-entry $\frac{\sin(j-i)}{\sin(j+i)}$. Prove

$$\det(A_{2n}) = \prod_{j<i}^{1,2n} \left(\frac{\sin(j-i)}{\sin(j+i)}\right)^2.$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA; Shalosh B. Ekhad, Rutgers University, New Brunswick, NJ, USA. As a first step, use $\sin(a\pm b)=\sin a\cos b \pm \sin b \cos b$ so that the claim is tantamount to

$$\det(A_{2n}) = \det\left(\frac{\tan j - \tan i}{\tan j + \tan i}\right)_{i,j}^{1,2n} = \prod_{j<i}^{1,2n} \left(\frac{\tan j - \tan i}{\tan j + \tan i}\right)^2.$$

We opt to generalize and use our popular technique of *Dodgson's Condensation formula.* Given an $n\times n$ matrix $\boldsymbol{M}$, let $\boldsymbol{M}_r(i,j)$ denote the $r\times r$ minor consisting of r contiguous rows and columns of $\boldsymbol{M}$ starting with row i and column j. In particular, $\boldsymbol{M}_n(1,1)=\det \boldsymbol{M}$. Then, according to Dodgson, there follows the recurrence $\boldsymbol{M}_n(1,1)\boldsymbol{M}_{n-2}(2,2)=\boldsymbol{M}_{n-1}(1,1)\boldsymbol{M}_{n-1}(2,2)-\boldsymbol{M}_{n-1}(2,1)\boldsymbol{M}_{n-1}(1,2)$. For the present purpose, consider (the claim) on the matrix determinants

$$\boldsymbol{M}_n(a+1,b+1) := \det\left(\frac{Cy_{j+b}-Dx_{i+a}}{y_{j+b}+x_{i+a}}\right)_{i,j}^{1,n}$$
$$= \frac{(C+D)^{n-1}(C\prod_j y_{j+b} + (-1)^n D \prod_i x_{i+a}) \prod_{i+a<j+b} (y_{j+b}-y_{i+b})(x_{j+a}-x_{i+a})}{\prod_{i,j}(y_{j+b}+x_{i+a})}.$$

However, one verifies this assertion routinely by checking Dodgson's recurrence is satisfied by the right-hand side, followed by comparing initial conditions (say, for $n=1$ and $n=2$). To get back to problem, let $a=b=0, C=D=1, y_j=\tan j, x_i=\tan i$. Obviously, if n is odd then the determinant vanishes. If $n\to 2n$ is even, we recover the desired solution to the proposer's determinantal evaluation. **Remark.** The fact that the right-hand side is perfect square is immediate from general principle because the matrix is skew-symmetric. □

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