SOLUTION TO PROBLEM #12304

Problem #12304. Proposed by M. Bataille (france). Let m and n be positive integers with $m < n$. Prove

$$\left(\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n-k}\right)\left(\sum_{k=0}^{m}\binom{n}{k}\frac{(-1)^k}{k+1}\right)=\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(n-k)(k+1)}.$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Replacing n by an indeterminate x, we intend to justify the equality between two rational functions (meromorphic functions with *simple poles* at $x = 0, 1, \dots, m$). That is to say,

$$\left(\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{x-k}\right)\left(\sum_{k=0}^{m}\binom{x}{k}\frac{(-1)^k}{k+1}\right)=\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(x-k)(k+1)}.$$

Clearly, these simple poles are shared by both sides. Hence, it suffices to compare the coefficients for $\frac{1}{x-j}$ for $j = 0, 1, \dots, m$. Fix such j. The claim, then, amounts to

$$\binom{m}{j}(-1)^j\sum_{k=0}^{m}\binom{j}{k}\frac{(-1)^k}{k+1}=\binom{m}{j}\frac{(-1)^j}{j+1}\qquad\Longleftrightarrow\qquad\sum_{k=0}^{j}\binom{j}{k}\frac{(-1)^k}{k+1}=\frac{1}{j+1}.$$

Starting with $\sum_{k=0}^{j}\binom{j}{k}(-1)^k x^k = (1-x)^j$, integrate both sides over the interval $0 \leq x \leq 1$ to get

$$\sum_{k=0}^{j}\binom{j}{k}(-1)^k\int_0^1 x^k\,dx=\sum_{k=0}^{j}\binom{j}{k}\frac{(-1)^k}{k+1}=\int_0^1(1-x)^j\,dx=\frac{1}{j+1}.$$

The proof is complete. $\square$

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