## SOLUTION TO PROBLEM #12305

*Problem #12305. Proposed by S. Sharma (India).* Let $\gamma$ be the Euler-Mascheroni constant. Prove

$$\int_0^1 \frac{x-1-x\log x}{x\log x - x\log^2 x}\,dx = \gamma.$$

*Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA.* First off, we have $\gamma = \int_0^1 \left(\frac{1}{\log x} + \frac{1}{1-x}\right) dx$ as one of the integral representations for $\gamma$. With this in mind, it suffices to show the difference between the two integral is zero. That means, evaluate

$$\begin{aligned}\int_0^1 \left(\frac{x-1-x\log x}{x\log x - x\log^2 x} - \frac{1}{\log x} - \frac{1}{1-x}\right) dx &= \int_0^1 \left(\frac{1}{\log x} - \frac{1}{x\log x - x\log^2 x} - \frac{1}{\log x} - \frac{1}{1-x}\right) dx \\ &= -\int_0^1 \left(\frac{1}{x\log x - x\log^2 x} + \frac{1}{1-x}\right) dx.\end{aligned}$$

So, we focus on last integral. This, however, goes as follows: since

$$\int \left(\frac{1}{x\log x - x\log^2 x} + \frac{1}{1-x}\right) dx = \log\left(\frac{\log x}{(1-x)(\log x - 1)}\right),$$

we compute two limits: $x \to 0^+$ and $x \to 1^-$. Both are executed via *L'Hôpital's Rule*, resulting in

$$\lim_{x\to 0^+} \log\left(\frac{\log x}{(1-x)(\log x - 1)}\right) = 0 \qquad \text{and} \qquad \lim_{x\to 1^-} \log\left(\frac{\log x}{(1-x)(\log x - 1)}\right) = 0.$$

The proof is complete. □

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