SOLUTION TO PROBLEM #12308

Problem #12308. Proposed by Cezar Lupu (China). What is the minimum value of $\int_0^1 (f'(x))^2 dx$ over all continuously differentiable functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 f(x)dx = \int_0^1 x^2 f(x)dx = 1$?

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Integration by parts gives $\int_0^1 f(x)dx = f(1) - \int_0^1 xf'(x)dx$ and $\int_0^1 x^2 f(x)dx = \frac{1}{3}f(1) - \frac{1}{3}\int_0^1 x^3 f'(x)dx$. Denote $c = f(1)$ as a free parameter and $g = f'$. Restate the problem as: *what is the minimum value of $\int_0^1 g(x)^2 dx$ over all continuous functions $g : [0,1] \to \mathbb{R}$ under the conditions $\int_0^1 xg(x)dx = c - 1$ and $\int_0^1 x^3 g(x)dx = c - 3$?*

Fix c. The problem is interpreted using the L^2 inner product $\langle F, G\rangle = \int_0^1 F(x)G(x)dx$. Specifically, we are given that $\langle x, g\rangle = c - 1$ and $\langle x^3, g\rangle = c - 3$, and we find the minimum value of $\langle g, g\rangle = \|g\|^2$. Since components of g orthogonal to both x and x^3 will only increase the norm of g, the minimizing function must lie in the subspace spanned by x and x^3. Suppose $g(x) = ax + bx^3$. Direct calculation leads to the system: $\frac{1}{5}b + \frac{1}{3}a = c - 1$ and $\frac{1}{7}b + \frac{1}{5}a = c - 3$ with solutions $a = 60 - \frac{15}{2}c, b = \frac{35}{2}c - 105$. Consequently, we obtain $\int_0^1 g(x)^2 dx = 10c^2 - 90c + 255$. Next, we minimize this quadratic expression which yields $c = \frac{9}{2}$ and thereby the required minimum becomes $\int_0^1 (f'(x))^2 dx = \frac{105}{2}$. $\square$

Proof (Calculus of variations). Integration by parts gives $\int_0^1 f(x)dx = f(1) - \int_0^1 xf'(x)dx$ and $\int_0^1 x^2 f(x)dx = \frac{1}{3}f(1) - \frac{1}{3}\int_0^1 x^3 f'(x)dx$. Denote $c = f(1)$ as a free parameter and $g = f'$. Restate the problem as: *what is the minimum value of $\int_0^1 g(x)^2 dx$ over all continuous functions $g : [0,1] \to \mathbb{R}$ under the conditions $\int_0^1 xg(x)dx = c - 1$ and $\int_0^1 x^3 g(x)dx = c - 3$?*

Fix c. As we have two constraints, $\int_0^1 f(x)dx = \int_0^1 x^2 f(x)dx = 1$, we introduce two Lagrange multipliers λ and μ, and attempt to optimize the functional

$$S[f] := \int_0^1 ((f'(x))^2 + \lambda f(x) + \mu x^2 f(x))dx.$$

Denote the integrand by $L(x, f(x), f'(x)) = (f'(x))^2 + \lambda f(x) + \mu x^2 f(x)$. A stationary point of the functional $S[f]$ satisfies the corresponding Euler-Lagrange equation, $\frac{\partial f}{\partial f} = \frac{d}{dx}\frac{\partial L}{\partial f'}$, which in the present case reduces to $\lambda + \mu x^2 = 2f''(x)$ or $f'(x) = k + \frac{1}{2}\lambda x + \frac{1}{6}\mu x^3$. Fix k. Direct calculation leads to the system: $b/30 + a/6 + k/2 = c - 1$ and $b/42 + a/10 + k/4 = c - 3$ with the resulting solutions $a = 120 - 45k/8 - 15c, b = -630 + 105k/8 + 105c$. Consequently, we obtain

$$\int_0^1 g(x)^2 dx = 10c^2 - 90c + \frac{9}{64}k^2 + 255.$$

We minimize this quadratic expression which yields $c = \frac{9}{2}$ and thereby $\int_0^1 (f'(x))^2 dx = \frac{105}{2} + \frac{9}{64}k^2$. The required minimum values turns out to be $\frac{105}{2}$. $\square$

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