SOLUTION TO PROBLEM #12317

Problem #12317. Proposed by S. Stewart (Saudi Arabia). Prove

$$\int_0^{\frac{\pi}{2}} \frac{\sin(4x)}{\log(\tan x)}\, dx = -14\frac{\zeta(3)}{\pi^2}.$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. Use $\sin(4x) = 2\sin(2x)\,\cos(2x)$ and $\tanh(\log(\tan x)) = -\cos(2x)$ then substitute $y = \cos(2x)$:

$$I := \int_0^{\frac{\pi}{2}} \frac{\sin(4x)}{\log(\tan x)}\, dx = -\int_0^{\frac{\pi}{2}} \frac{2\sin(2x)\,\cos(2x)}{\tanh^{-1}(\cos(2x))}\, dx = -2\int_0^1 \frac{y}{\tanh^{-1} y}\, dy.$$

Now, letting $y = \tanh u$ results in $I = -2\int_0^\infty \frac{\tanh u \,\mathrm{sech}^2 u}{u}\, du$. Integrate by parts with $a = \frac{1}{u}$ and $b' = \tanh u \,\mathrm{sech}^2 u$ so that $I = -\int_0^\infty \frac{\tanh^2 u}{u^2}\, du$. At this point, we invoke the Weierstrass product formula $\cosh(u) = \prod_{k=0}^\infty \left(1 + \frac{4u^2}{\pi^2(2k+1)^2}\right)$ and differentiate to find that $\frac{\tanh u}{u} = \sum_{k=0}^\infty \frac{8}{\pi^2(2k+1)^2+4u^2}$. Based on the (simple) fact $\int_0^\infty \frac{dx}{\alpha^2+x^2} = \frac{\pi}{2\alpha}$, one obtains $\int_0^\infty \frac{dx}{(\alpha^2+x^2)(\beta^2+x^2)} = \frac{\pi}{2\alpha\beta(\alpha+\beta)}$. Therefore,

$$\begin{aligned}
\int_0^\infty \frac{\tanh^2 u}{u^2}\, du &= \int_0^\infty \sum_{n,m\geq 0} \frac{64\, du}{(\pi^2(2n+1)^2+4u^2)\,(\pi^2(2m+1)^2+4u^2)} \\
&= \sum_{n,m\geq 0} \int_0^\infty \frac{64\, du}{(\pi^2(2n+1)^2+4u^2)\,(\pi^2(2m+1)^2+4u^2)} \\
&= \frac{16}{\pi^2} \sum_{n,m\geq 0} \frac{1}{(2n+1)(2m+1)(2n+1+2m+1)} \\
&= \frac{16}{\pi^2} \sum_{n,m\geq 0} \frac{1}{(2n+1)(2m+1)} \int_0^1 t^{2n+1+2m+1}\frac{dt}{t} \\
&= \frac{4}{\pi^2} \int_0^1 \log^2\left(\frac{1-t}{1+t}\right) \frac{dt}{t}.
\end{aligned}$$

Making the substitution $w = \frac{1-t}{1+t}$ yields

$$\int_0^1 \log^2\left(\frac{1-t}{1+t}\right) dt = 2\int_0^1 \frac{\log^2 w}{1-w^2}\, dw = 2\int_0^1 \log^2 w \sum_{j=0}^\infty w^{2j}\, dw = \sum_{j=0}^\infty \frac{4}{(2j+1)^3} = \frac{7}{2}\zeta(3).$$

Combining all these calculations we arrive at $I = -14\frac{\zeta(3)}{\pi^2}$. $\square$