SOLUTION TO PROBLEM #12322

Problem #12322. Proposed by A. Dzhumadil'daev (Kazakhstan). Given real numbers $x_1, \dots, x_{2n}$, let A be the skew-symmetric matrix with entries $a_{i,j} = (x_j - x_i)^2$ for $1 \le i < j \le 2n$. Prove

$$\det A_{2n} = 4^{n-1}(x_1 - x_2)^2(x_2 - x_3)^2 \cdots (x_{2n-1} - x_{2n})^2(x_{2n} - x_1)^2. \tag{1}$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Let sgn be the *sign function* defined as: $\operatorname{sgn}(u) = 1$ if $u > 0$; $\operatorname{sgn}(u) = -1$ if $u < 0$; $\operatorname{sgn}(0) = 0$. Then, the entries of A can be given by $a_{i,j} = \operatorname{sgn}(j-i)\cdot(x_j - x_i)^2$, for $1 \le i, j \le 2n$. It is clear that $\det A_{2n}$ vanishes if $x_i = x_{i+1}$ for each i, with the convention that $x_{2n+1} = x_1$. Hence $(x_1 - x_2)\cdots(x_{2n} - x_1)$ is factor of $\det A$. On the other hand, since A_{2n} is skew-symmetric, every factor must be a perfect square. So, indeed, $(x_1 - x_2)^2 \cdots (x_{2n} - x_1)^2$ is factor of $\det A_{2n}$. Since both sides of equation (1) are polynomials of degree $4n$, it remains to show that they share the same constant factor. To determine this constant, choose $x_k = 0$ for k even and $x_k = 1$ for k odd. The right-hand side of (1) yields 4^{n-1}. In the upper triangle, the entries of A_{2n} alternate values between 0 and 1 while in the lower triangle they alternate between 0 and -1, with 0 diagonals.

Now, add the last row to the even-numbered rows and subtract the penultimate row from the odd-numbered rows. To illustrate the procedure, take for example $2n = 6$. Then

$$\det A_6 = \det \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix} = \det \begin{pmatrix} 0 & 2 & 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

Clearly $\det A_2 = 1$. For $2n > 2$, following the above procedure, proceed with Laplace expansion by the last two columns, successively, reducing the size of the matrices until one reaches the determinant of a 2×2 matrix. The result is $\det A_{2n} = 4^{n-1}$. The proof is now complete. $\square$