SOLUTION TO PROBLEM #12324

Problem #12324. Proposed by A. Stadler (Switzerland). Let a and b be positive real numbers. Prove

$$\int_0^\infty \frac{dx}{\sqrt{ax^4+2(2b-a)x^2+a}} = \int_0^\infty \frac{dx}{\sqrt{bx^4+2(2a-b)x^2+b}}.$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Make the substitution $x=\tan\theta, dx=\sec^2\theta\, d\theta$ and later simplify with $\tan\theta=\frac{\sin\theta}{\cos\theta}$. Next, make another substitution $\theta=\varphi+\frac{\pi}{4}, d\theta=d\varphi$. So, the integral on the left-hand side takes the form

$$\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a\sin^4\theta+2(2b-a)\sin^2\theta\cos^2\theta+a\cos^4\theta}} \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{d\varphi}{\sqrt{a\sin^4(\varphi+\frac{\pi}{4})+2(2a-b)\sin^2(\varphi+\frac{\pi}{4})\cos^2(\varphi+\frac{\pi}{4})+a\cos^4(\varphi+\frac{\pi}{4})}} \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{d\varphi}{\sqrt{b\sin^4\varphi+2(2a-b)\sin^2\varphi\cos^2\varphi+b\cos^4\varphi}}.
\end{aligned}$$

A similar argument applies to the integral on the right-hand side:

$$I_2 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{d\varphi}{\sqrt{a\sin^4\varphi+2(2b-a)\sin^2\varphi\cos^2\varphi+a\cos^4\varphi}}.$$

At this point, if we let $\varphi=t+\frac{\pi}{4}, d\varphi=dt$ and since $\sin^2(t+\frac{\pi}{2})=\cos t, \cos^2(t+\frac{\pi}{2})=\sin^2 t$ then

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \frac{d\varphi}{\sqrt{a\sin^4(\varphi+\frac{\pi}{4})+2(2a-b)\sin^2(\varphi+\frac{\pi}{4})\cos^2(\varphi+\frac{\pi}{4})+a\cos^4(\varphi+\frac{\pi}{4})}} \\
&= \int_{-\frac{\pi}{4}}^{0} \frac{dt}{\sqrt{a\sin^4(t+\frac{\pi}{2})+2(2a-b)\sin^2(t+\frac{\pi}{2})\cos^2(t+\frac{\pi}{2})+a\cos^4(t+\frac{\pi}{2})}} \\
&= \int_{-\frac{\pi}{4}}^{0} \frac{dt}{\sqrt{a\sin^4 t+2(2a-b)\sin^2 t\cos^2 t+a\cos^4 t}}.
\end{aligned}$$

Analogusly, if $\varphi=t-\frac{\pi}{4}$ and $d\varphi=dt$ then

$$\begin{aligned}
&\int_{-\frac{\pi}{4}}^{0} \frac{d\varphi}{\sqrt{a\sin^4(\varphi+\frac{\pi}{4})+2(2a-b)\sin^2(\varphi+\frac{\pi}{4})\cos^2(\varphi+\frac{\pi}{4})+a\cos^4(\varphi+\frac{\pi}{4})}} \\
&= \int_0^{\frac{\pi}{4}} \frac{dt}{\sqrt{a\sin^4 t+2(2a-b)\sin^2 t\cos^2 t+a\cos^4 t}}.
\end{aligned}$$

Therefore, we conclude that $I_1=I_2$ as desired. The proof is complete. □

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