SOLUTION TO PROBLEM #12327

Problem #12327. Proposed by M. Merca (Romania). For $n \geq 0$, prove

$$\sum_{k=0}^{n} \binom{n}{k}_{q^2} q^k = \sum_{k=0}^{n} \binom{n}{k}_{q^2} q^{k(k-1)+(n-k)^2-n(n-1)/2}.$$

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Let $f_n(q) = \sum_{k=0}^{n} \binom{n}{k}_{q^2} q^k$ and $g_n(q) = \sum_{k=0}^{n} \binom{n}{k}_{q^2} q^{k(k-1)+(n-k)^2-n(n-1)/2}$. We make use of the recurrence $\binom{n+1}{k}_{q^2} = \binom{n}{k}_{q^2} + q^{2n+2-2k}\binom{n}{k-1}_{q^2}$, the symmetry $\binom{n}{k}_{q^2} = \binom{n}{n-k}_{q^2}$ followed by the replacement $k \to n-k$ (in the sum) so that

$$\begin{aligned} f_{n+1}(q) &= \sum_{k=0}^{n+1} \binom{n}{k}_{q^2} q^k + \binom{n}{k-1}_{q^2} q^{2n+2-k} = \sum_{k=0}^{n} \binom{n}{k}_{q^2} q^k + q^{n+1} \sum_{k=0}^{n} \binom{n}{k}_{q^2} q^{n-k} \\ &= f_n(q) + q^{n+1} \sum_{k=0}^{n} \binom{n}{n-k}_{q^2} q^{n-k} = (1+q^{n+1}) f_n(q). \end{aligned}$$

Utilizing the recurrence $\binom{n+1}{k}_{q^2} = q^{2k}\binom{n}{k}_{q^2} + \binom{n}{k-1}_{q^2}$ instead, we obtain

$$\begin{aligned} g_{n+1}(q) &= \sum_{k=0}^{n+1} \binom{n}{k}_{q^2} q^{k(k+1)+(n+1-k)^2-n(n+1)/2} + \binom{n}{k-1}_{q^2} q^{k(k-1)+(n+1-k)^2-n(n+1)/2} \\ &= q^{n+1} \sum_{k=0}^{n} \binom{n}{k}_{q^2} q^{k(k-1)+(n-k)^2-n(n-1)/2} + \sum_{k=0}^{n} \binom{n}{k}_{q^2} q^{k(k+1)+(n-k)^2-n(n+1)/2} \\ &= q^{n+1} g_n(q) + \sum_{k=0}^{n} \binom{n}{n-k}_{q^2} q^{(n-k)(n-k+1)+k^2-n(n+1)/2} \\ &= q^{n+1} g_n(q) + \sum_{k=0}^{n} \binom{n}{k}_{q^2} q^{k(k-1)+(n-k)^2-n(n-1)/2} = (q^{n+1}+1) g_n(q). \end{aligned}$$

Thus, $f_{n+1}(q) = (1+q^{n+1}) f_n(q), g_{n+1}(q) = (1+q^{n+1}) g_n(q)$. On the other hand, it is clear that $f_0(q) = g_0(q) = 1$. We arrive at the desired equality. Incidentally, the recurrence reveals both sum in the problem evaluate in a closed form as $(-q;q)_n = (1+q)(1+q^2)\cdots(1+q^n)$, for $n \geq 1$. $\square$