SOLUTION TO PROBLEM #12332

Problem #12332. Proposed by F. Holland (Ireland). Prove

$$\int_0^\infty \frac{\tanh^2 x}{x^2}\,dx = \frac{14\zeta(3)}{\pi^2}.$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Let $I := \int_0^\infty \frac{\tanh^2 x}{x^2}\,dx = \int_{-\infty}^\infty \frac{\tanh^2 x}{2x^2}\,dx$. First substitute $y = e^{-2x}$ and then integrate by parts using $u = \left(\frac{1-y}{1+y}\right)^2$ and $dv = \frac{dy}{y\log^2 y}$. The result is then

$$I = \int_0^\infty \left(\frac{1-y}{1+y}\right)^2 \frac{dy}{y\log^2 y} = 4\int_0^\infty \frac{y-1}{(1+y)^3}\frac{dy}{\log y}.$$

Define $Q(b) := 4\int_0^\infty \frac{y^b-1}{(1+y)^3}\frac{dy}{\log y}$ so that $\frac{d}{db}Q(b) = 4\int_0^\infty \frac{y^b}{(1+y)^3}dy = \frac{2\pi\, b(1-b)}{\sin \pi b}$. Now, integrate by parts with $u = 2\pi b(1-b)$ and $dv = \csc(\pi b)$ followed by the substitution $\sqrt{z} = \tan(\frac{\pi b}{2})$. Thus

$$I = \int_0^1 \frac{dQ}{db}\,db = \int_0^1 (4b-2)\log\left(\tan\frac{\pi b}{2}\right)db = \int_0^1 4b\,\log\left(\tan\frac{\pi b}{2}\right)db = \int_0^\infty \frac{4\log z\cdot\tan^{-1}\sqrt{z}}{\pi^2\sqrt{z}\,(1+z)}\,dz.$$

Let $R(c) := \int_0^\infty \frac{\log z\cdot\tan^{-1}(c\sqrt{z})}{\sqrt{z}\,(1+z)}\,dz$ so that $\frac{d}{dc}R(c) = \int_0^\infty \frac{\log z}{(1+z)(1+c^2z)}\,dz = \frac{2\log^2 c}{1-c^2}$ and hence

$$I = \frac{4}{\pi^2}\int_0^1 \frac{d}{dc}R(c)\,dc = \frac{8}{\pi^2}\int_0^1 \frac{\log^2 c}{1-c^2}\,dc = \frac{8}{\pi^2}\cdot\frac{7\zeta(3)}{4} = \frac{14\zeta(3)}{\pi^2}. \qquad \square$$