SOLUTION TO PROBLEM #12337

Problem #12337. Proposed by H. Ohtsuka (Japan). For $k \in \{0,1,2\}$, let

$$S_k = \sum \frac{(-4)^n}{2n+1}\binom{2n}{n}^{-1},$$

where the sum is taken over all non-negative integers n that congruent to k modulo 3. Prove
(a) $S_0 = \frac{\ln(1+\sqrt{2})}{3\sqrt{2}} + \frac{\pi}{6}$,
(b) $S_1 = \frac{\ln(1+\sqrt{2})}{3\sqrt{2}} - \frac{\ln(2+\sqrt{3})}{2\sqrt{3}} - \frac{\pi}{12}$,
(c) $S_2 = \frac{\ln(1+\sqrt{2})}{3\sqrt{2}} + \frac{\ln(2+\sqrt{3})}{2\sqrt{3}} - \frac{\pi}{12}$.

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Euler's *beta function* gives $\int_0^1 (x(1-x))^m\,dx = \frac{1}{(2m+1)\binom{2m}{m}}$. Apply this and employ and factor $1+64x^3(1-x)^3 = -((2x-1)^2-2)((2x-1)^4-(2x-1)^2+1)$ and substitute $u=2x-1$. Thus

$$\begin{aligned}
S_0 &= \sum_{n\geq 0}(-4)^{3n}\int_0^1 (x(1-x))^{3n}dx = \int_0^1 \frac{1}{1+64x^3(1-x)^3}\,dx,\\
&= -\frac{1}{3}\int_0^1\left(\frac{1}{(2x-1)^2-2} - \frac{(2x-1)^2+1}{(2x-1)^4-(2x-1)^2+1}\right)dx\\
&= -\frac{1}{6}\int_{-1}^1\left(\frac{1}{u^2-2} - \frac{u^2+1}{u^4-u^2+1}\right)du = -\frac{1}{3}\int_0^1\left(\frac{1}{u^2-2} - \frac{1}{u^2+1} - \frac{3u^2}{u^6+1}\right)du\\
&= -\frac{1}{3}\int_0^1\left(\frac{1}{u^2-2} - \frac{1}{u^2+1}\right)du + \frac{1}{3}\int_0^1 \frac{1}{y^2+1}\,dy = \frac{\ln(1+\sqrt{2})}{3\sqrt{2}} + \frac{\pi}{6}.
\end{aligned}$$

Analogously, we proceed with the remaining evaluations (some integrals being standard)

$$\begin{aligned}
S_1 &= \int_0^1 \frac{-4x(1-x)}{1+64x^3(1-x)^3}\,dx = -\frac{1}{3}\int_0^1\left(\frac{1}{(2x-1)^2-2} - \frac{(2x-1)^2-2}{(2x-1)^4-(2x-1)^2+1}\right)dx\\
&= -\frac{1}{3}\int_0^1\left(\frac{1}{u^2-2} - \frac{1}{u^2+1} + \frac{3}{u^6+1}\right)du = \frac{\ln(1+\sqrt{2})}{3\sqrt{2}} - \frac{\ln(2+\sqrt{3})}{2\sqrt{3}} - \frac{\pi}{12},
\end{aligned}$$

$$\begin{aligned}
S_2 &= \int_0^1 \frac{16x^2(1-x)^2}{1+64x^3(1-x)^3}\,dx = -\frac{1}{3}\int_0^1\left(\frac{((2x-1)^2-1)^2}{(2x-1)^2-2} - \frac{((2x-1)^2+1)((2x-1)^2-1)^2}{(2x-1)^4-(2x-1)^2+1}\right)dx\\
&= -\frac{1}{3}\int_0^1\left(\frac{1}{u^2-2} + \frac{2}{u^2+1} + \frac{3u^2}{u^6+1} - \frac{3}{u^6+1}\right)du = \frac{\ln(1+\sqrt{2})}{3\sqrt{2}} + \frac{\ln(2+\sqrt{3})}{2\sqrt{3}} - \frac{\pi}{12}. \qquad \square
\end{aligned}$$

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