SOLUTION TO PROBLEM #12338

Problem #12338. Proposed by I. Mezo (China). Prove

$$\int_0^\infty \frac{\cos x - 1}{x(e^x - 1)}\,dx = \frac{1}{2}\ln\left(\frac{\pi}{\sinh \pi}\right).$$

Solution by Tewodros Amdeberhan and Victor H Moll, Tulane University, New Orleans, LA, USA. Start by rewriting $\int_0^\infty \frac{\cos x-1}{x(e^x-1)}\,dx = \int_0^\infty \frac{\cos x-1}{xe^x(1-e^{-x})}\,dx = \sum_{n\geq 0}\int_0^\infty \frac{(\cos x-1)e^{-(n+1)x}}{x}\,dx$. Define the function $I(a) := \int_0^\infty \frac{(\cos x-1)e^{-ax}}{x}\,dx$ so that $\frac{d}{da}I(a) = \int_0^\infty (1-\cos x)e^{-ax}\,dx = \frac{1}{a(a^2+1)}$. Therefore, $I(a) = \frac{1}{2}\ln\left(\frac{a^2}{1+a^2}\right) + C$. Taking the limit $a \to \infty$ resolves $C = 0$ and hence $I(a) = \frac{1}{2}\ln\left(\frac{a^2}{1+a^2}\right)$. So,

$$\int_0^\infty \frac{\cos x - 1}{x(e^x - 1)}\,dx = \frac{1}{2}\sum_{n=0}^{\infty}\ln\left(\frac{(n+1)^2}{1+(n+1)^2}\right) = \frac{1}{2}\ln\left(\prod_{k=1}^{\infty}\frac{k^2}{1+k^2}\right).$$

On the other hand, we recall the infinite product expansion $\frac{\sinh z}{z} = \prod_{k=1}^\infty \left(1 + \frac{z^2}{\pi^2 k^2}\right)$ which yields

$$\int_0^\infty \frac{\cos x - 1}{x(e^x - 1)}\,dx = \frac{1}{2}\ln\left(\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)^{-1}\right) = \frac{1}{2}\ln\left(\frac{\pi}{\sinh \pi}\right). \qquad \square$$

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