SOLUTION TO PROBLEM #12349

Problem #12349. Proposed by R. Tauraso (Italy). Let A_n be the set of permutations of $\{1,\dots,n\}$ that have at least one fixed point. For $\pi\in A_n$, we write $Fix(\pi)$ for $\{j:\pi(j)=j\}$. Evaluate

$$a_n:=\sum_{\pi\in A_n}\left(\frac{sgn(\pi)}{|Fix(\pi)|}\sum_{j\in Fix(\pi)}j\right).$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Let $D(n)$ be the set of derangements on n letters. We rearrange the sum by the number k of fixed points $i_1,\dots,i_k$:

$$\begin{aligned}
a_n&=\sum_{k=1}^{n}\sum_{\pi\in D(n-k)}\frac{sgn(\pi)}{k}\sum_{1\le i_1<\cdots<i_k\le n}(i_1+\cdots+i_k)\\
&=\frac{1}{n}\binom{n+1}{2}+\sum_{k=1}^{n-2}\sum_{\pi\in D(n-k)}\frac{sgn(\pi)}{k}\sum_{1\le i_1<\cdots<i_k\le n}(i_1+\cdots+i_k)\\
&=\frac{n+1}{2}+\sum_{k=1}^{n-2}\frac{(-1)^{n-1-k}(n-1-k)}{k}\sum_{1\le i_1<\cdots<i_k\le n}(i_1+\cdots+i_k)\\
&=\frac{n+1}{2}+\binom{n+1}{2}\sum_{k=1}^{n-2}\frac{(-1)^{n-1-k}(n-1-k)}{k}\binom{n-1}{k-1}\\
&=\frac{n+1}{2}+\binom{n+1}{2}\left[\frac{((-1)^n+1)(n-1)}{n}-1\right]\\
&=\frac{(-1)^n(n^2-1)}{2};
\end{aligned}$$

where we noted that $\boldsymbol{\sum_{\pi\in D(n)} sgn(\pi)=(-1)^{n-1}(n-1)}$ and if $|Fix(\pi)|=n-1$ then $|Fix(\pi)|=n$, also that $\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}=0$. $\square$

Lemma. $\boldsymbol{\sum_{\pi\in D(n)} sgn(\pi)=(-1)^{n-1}(n-1)}$.

Proof. We may interpret the sum as the determinant of an $n\times n$ matrix A_n with zeros on the main diagonal and ones everywhere else. Then A_n+I_n is the matrix consisting entirely of ones, which clearly has $n-1$ zero rows after row-reduction. Therefore A_n has eigenvalue -1, repeated (at least) $n-1$ times, and since $trace(A_n)=0$, the other eigenvalue is $n-1$. So, their product gives $\det(A_n)=(-1)^{n-1}(n-1)$. $\square$

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