SOLUTION TO PROBLEM #12351

Problem #12351. Proposed by S. Stewart (Saudi Arabia). Evaluate

$$I := \int_0^\infty \frac{\log(\cos^2 x)\cdot\sin^3 x}{x^3\cdot(1+2\cos^2 x)}\,dx.$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. We strat with two facts: the Poisson integral $\sum_{n=-\infty}^{\infty}\frac{1}{(y+n)^2}=\frac{\pi^2}{\sin^2(\pi y)}$ and the identity $2\cos^4\theta+2\sin^4\theta=1+\cos^2(2\theta)$. We note the integrand is an even function and hence

$$I=\frac{1}{2}\sum_{k=-\infty}^{\infty}\int_0^\pi \frac{(-1)^k}{(t+k\pi)^3}\frac{\log(\cos^2 t)\cdot\sin^3 t}{1+2\cos^2 t}dt=\frac{1}{2}\int_0^\pi\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{(t+k\pi)^3}\frac{\log(\cos^2 t)\cdot\sin^3 t}{1+2\cos^2 t}dt.$$

Differentiating the above Poisson integral, $\sum_{-\infty}^{\infty}\frac{1}{(y+n)^3}=\pi^3\cot(\pi y)\csc^2(\pi y)$ and consequently $\sum_{-\infty}^{\infty}\left(\frac{1}{(t+2k\pi)^3}-\frac{1}{(t+(2k+1)\pi)^3}\right)=\frac{1}{8}\cot(\frac{t}{2})\csc^2(\frac{t}{2})+\frac{1}{8}\tan(\frac{t}{2})\sec^2(\frac{t}{2})=\frac{1+\cos^2 t}{2\sin^3 t}$. Therefore, our integral becomes $I=\frac{1}{4}\int_0^\pi\frac{\log(\cos^2 t)\cdot(1+\cos^2 t)}{1+2\cos^2 t}dt=\frac{1}{2}\int_0^{\pi/2}\frac{\log(\cos^2 t)\cdot(1+\cos^2 t)}{1+2\cos^2 t}dt$. Substitute $z=\tan t$ and $dz=\sec^2 t\,dt$ to find $I=-\frac{1}{2}\int_0^\infty\frac{\log(1+z^2)\cdot(2+z^2)}{(1+z^2)(3+z^2)}dz=-\frac{1}{4}I_1-\frac{1}{12}I_2$. Substitute $z=\sqrt{w}$ so that

$$I_1=\int_0^\infty\frac{\log(1+z^2)\,dz}{1+z^2}=\int_0^\infty\frac{\log(1+w)\,dw}{2\sqrt{w}\,(1+w)}\qquad\text{and}\qquad I_2=\int_0^\infty\frac{\log(1+z^2)\,dz}{1+\frac{1}{3}z^2}.$$

The derivative $\frac{d}{dn}$ of the Euler's Beta function $\int_0^\infty\frac{dw}{\sqrt{w}\,(1+w)^n}=\frac{\Gamma(\frac{1}{2})\,\Gamma(n-\frac{1}{2})}{\Gamma(n)}$ leads to the integral $-\int_0^\infty\frac{\log(1+w)\,dw}{\sqrt{w}\,(1+w)^n}=\frac{\Gamma(\frac{1}{2})\Gamma(n-\frac{1}{2})[\psi(n-\frac{1}{2})-\psi(n)]}{\Gamma(n)}$. Thus, $\int_0^\infty\frac{\log(1+w)\,dw}{\sqrt{w}\,(1+w)}=\frac{-\Gamma(\frac{1}{2})^2[\psi(\frac{1}{2})-\psi(1)]}{\Gamma(1)}=2\pi\log 2$. That means $I_1=\pi\log 2$. If $J(a):=\int_0^\infty\frac{\log((1+\frac{1}{3}z^2)a+\frac{2}{3}z^2)\,dz}{1+\frac{1}{3}z^2}$ then $\frac{dJ}{da}=\int_0^\infty\frac{dz}{\frac{1}{3}(2+a)z^2+a}=\frac{\pi\sqrt{3}}{2\sqrt{a}\sqrt{2+a}}$. Now, $I_2=J(1)=J(1)-J(0)+J(0)=\int_0^1 J'(a)\,da+J(0)=\frac{\pi\sqrt{3}}{2}\int_0^1\frac{da}{\sqrt{a}\sqrt{2+a}}+\int_0^\infty\frac{\log(\frac{2}{3}z^2)\,dz}{1+\frac{1}{3}z^2}$. But,

$$\int_0^1\frac{da}{\sqrt{a}\sqrt{2+a}}=\log(1+a+\sqrt{a}\sqrt{2+a})|_0^1=\log(2+\sqrt{3}).$$

On the other hand, the substitution $z=\sqrt{3w}$ implies $J(0)=\frac{\sqrt{3}}{2}\int_0^\infty\frac{(\log 2+\log w)dw}{\sqrt{w}(1+w)}$. Again, working with Euler's Beta, we obtain $\int_0^\infty\frac{\log 2\,dw}{\sqrt{w}(1+w)}=\pi\log 2$. From $\frac{d}{dm}\int_0^\infty\frac{w^{m-1}dw}{1+w}=\frac{d}{dm}\Gamma(m)\Gamma(1-m)$, we gather that $\int_0^\infty\frac{w^{m-1}\log(w)\,dw}{1+w}=\Gamma(m)\Gamma(1-m)[\psi(m)-\psi(1-m)]$. So, $\int_0^\infty\frac{\log w\,dw}{\sqrt{w}(1+w)}=0$. Combining all of the above results in $I=-\frac{1}{4}I_1-\frac{1}{12}I_2=-\frac{1}{4}\pi\log 2-\frac{\pi\sqrt{3}}{24}\log(2+\sqrt{3})-\frac{1}{24}\pi\sqrt{3}\log 2$. To be concise, we may write

$$\int_0^\infty\frac{\log(\cos^2 x)\cdot\sin^3 x}{x^3\cdot(1+2\cos^2 x)}\,dx=-\frac{\pi\log 2}{4}-\frac{\pi\sqrt{3}\log(1+\sqrt{3})}{12}.\qquad\square$$