SOLUTION TO PROBLEM #12360

Problem #12360. Proposed by D. M. Batinetu-Giurgiu (Romania). Evaluate

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{x_{n+1}}-\frac{n^2}{x_n}\right),$$

where $x_n=\sqrt[n]{\sqrt[1]{1!}\sqrt[2]{2!}\sqrt[3]{3!}\cdots\sqrt[n]{n!}}$.

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Let $y_n:=\frac{n^{2n}}{x_n^n}$.

$$\lim_{n\to\infty}\frac{y_{n+1}}{n\,y_n}=\lim_{n\to\infty}\frac{(n+1)^{2n+2}}{n^{2n+1}\sqrt[n+1]{(n+1)!}}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n+1}\cdot\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}.$$

It's easy to check $\lim_n\left(1+\frac{1}{n}\right)^{2n+1}=e^2$. On the other hand, if $z_n=\frac{n}{\sqrt[n]{n!}}$ then

$$\lim_{n\to\infty}\log z_n=\lim_{n\to\infty}\frac{\log\left(\frac{n^n}{n!}\right)}{n}=\lim_{n\to\infty}\frac{\log\left(\frac{(n+1)^{n+1}}{(n+1)!}\right)-\log\left(\frac{n^n}{n!}\right)}{n+1-n}=\lim_{n\to\infty}\log\left(\frac{n+1}{n}\right)^n=1$$

where we invoked Stolz-Cesáro's Theorem. So, $\lim_{n\to\infty}\frac{y_{n+1}}{n\,y_n}=e^2\cdot e=e^3$. By an application pf the Cauchy-D'Alembert limits, we gather

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{y_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{y_n}}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{y_{n+1}}\cdot\frac{y_n}{n^n}=\lim_{n\to\infty}\frac{n\,y_n}{y_{n+1}}\left(1+\frac{1}{n}\right)^{n+1}=e^{-3}e=e^{-2}.$$

Denote $w_n:=\sqrt[n+1]{y_{n+1}}-\sqrt[n]{y_n}=\frac{(n+1)^2}{x_{n+1}}-\frac{n^2}{x_n}$ and $u_n:=\frac{\sqrt[n+1]{y_{n+1}}}{\sqrt[n]{y_n}}$. We proceed to compute

$$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{\sqrt[n+1]{y_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{y_n}}\cdot\frac{n+1}{n}=1,$$
$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{y_{n+1}}{y_n}\,\frac{1}{\sqrt[n+1]{y_{n+1}}}\right)=\lim_{n\to\infty}\left(\frac{y_{n+1}}{n\,y_n}\left(\frac{n}{n+1}\right)\frac{n+1}{\sqrt[n+1]{y_{n+1}}}\right)=e.$$

By L'Hôpital's rule, $\lim_{x\to1}\frac{x-1}{\log x}=1$ implying $\lim_{n\to\infty}\frac{u_n-1}{\log u_n}=1$. Finally,

$$\lim_{n\to\infty}w_n=\lim_{n\to\infty}\sqrt[n]{y_n}(u_n-1)=\lim_{n\to\infty}\frac{\sqrt[n]{y_n}}{n}\cdot\frac{u_n-1}{\log u_n}\cdot\log u_n^n=w^2\cdot1\cdot1=e^2.\qquad\square$$

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