SOLUTION TO PROBLEM #12361

Problem #12361. Proposed by H. Ohtsuka (Japan). For a non-negative integer k, assume $r_{3k}=0, r_{3k+1}=1$, and $r_{3k+2}=-1$. Prove

$$\sum_{k=0}^{n-1}\binom{2k}{k}=\sum_{k=0}^{n}r_k\binom{2n}{n-k}$$

for every positive integer n.

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Denote the left-hand side by f_n and the right-hand side by g_n. So, $f_{n+1}-f_n=\binom{2n}{n}$ while

$$\begin{aligned}
g_{n+1}-g_n &= \sum_{k\geq 0}\binom{2n+2}{n-3k}+\sum_{k\geq 0}\binom{2n}{n-3k-2}-\sum_{k\geq 0}\binom{2n+2}{n-3k-1}-\sum_{k\geq 0}\binom{2n}{n-3k-1} \\
&= \sum_{k\geq 0}\binom{2n}{n-3k}+2\sum_{k\geq 0}\binom{2n}{n-3k-1}+2\sum_{k\geq 0}\binom{2n}{n-3k-2} \\
&\quad -2\sum_{k\geq 0}\binom{2n}{n-3k-1}-2\sum_{k\geq 0}\binom{2n}{n-3k-2}-\sum_{k\geq 0}\binom{2n}{n-3k-3} \\
&= \sum_{k\geq 0}\binom{2n}{n-3k}-\sum_{k\geq 0}\binom{2n}{n-3k-3}=\binom{2n}{n}
\end{aligned}$$

where we have utilized the binomial recurrence (twice), i.e. $\binom{a}{b}=\binom{a-2}{b}+2\binom{a-2}{b-1}+\binom{a-2}{b-2}$. Thus $f_{n+1}-f_n=g_{n+1}-g_n$. It is easy to check that $f_1=g_1$. The assertion is immediate. $\square$

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