SOLUTION TO PROBLEM #12364

Problem #12364. Proposed by R. Tauraso (Italy). Let n be a positive integer, and let z be a complex number not in $\{-1,-2,\dots,-n\}$. Denote $\binom{\alpha}{k}=\frac{1}{k!}\prod_{i=0}^{k-1}(\alpha-i)$. Prove

$$\sum_{1\le j\le k\le n}\frac{(-1)^{k-1}}{(z+j)\,k^2}\binom{z+n}{n-k}=\binom{z+n}{n}\sum_{1\le j\le k\le n}\frac{1}{(z+j)^2k}.$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Rewrite the problem:

$$\sum_{j=1}^{n}\frac{1}{(z+j)^2\prod_{i=1}^{j-1}(z+i)}\sum_{k=j}^{n}\frac{(-1)^{k-1}}{k^2(n-k)!}\frac{n!}{\prod_{i=j+1}^{k}(z+i)}=\sum_{j=1}^{n}\frac{1}{(z+j)^2}\sum_{k=j}^{n}\frac{1}{k}. \tag{1}$$

Clearly, both sides are meromorphic functions with poles of order two exactly at $\{-1,-2,\dots,-n\}$. So, for $1\le\ell\le n$, multiply through by $(z+\ell)^2$ and compute the values at $z=-\ell$ to compare

$$\sum_{k=\ell}^{n}\frac{(-1)^{k-\ell}}{k}\binom{k-1}{\ell-1}\binom{n}{k}=\sum_{k=\ell}^{n}\frac{1}{k}. \tag{2}$$

Fix ℓ and induct on $n\ge\ell$. The base case $n=\ell$ is trivial (both sides on (2) equal $\frac{1}{\ell}$). Assume (2) holds for n. For $n+1$, we use Pascal's recurrence and the induction hypothesis to the effect

$$\begin{aligned}\sum_{k=\ell}^{n+1}\frac{(-1)^{k-\ell}}{k}\binom{k-1}{\ell-1}\binom{n+1}{k}&=\sum_{k=\ell}^{n+1}\frac{(-1)^{k-\ell}}{k}\binom{k-1}{\ell-1}\binom{n}{k}+\sum_{k=\ell}^{n+1}\frac{(-1)^{k-\ell}}{k}\binom{k-1}{\ell-1}\binom{n}{k-1}\\&=\sum_{k=\ell}^{n}\frac{1}{k}+\frac{1}{n+1}\sum_{k=\mathbf{1}}^{n+1}(-1)^{k-\ell}\binom{k-1}{\ell-1}\binom{n+1}{k}.\end{aligned}$$

Since $\sum_{k=\mathbf{1}}^{n+1}(-1)^k k^m\binom{n+1}{k}=0$ if $m>0$ while $\sum_{k=\mathbf{1}}^{n+1}(-1)^{k-1}\binom{n+1}{k}=1$ when $m=0$, we gather that $\sum_{k=\ell}^{n+1}\frac{(-1)^{k-\ell}}{k}\binom{k-1}{\ell-1}\binom{n+1}{k}=\sum_{k=\ell}^{n}\frac{1}{k}+\frac{1}{n+1}=\sum_{k=\ell}^{n+1}\frac{1}{k}$. Thus (2) holds. Consequently, both sides of equation (1) completely agree in their singularities. It remains to check (1) at, say $z=0$; that is,

$$\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}\frac{(-1)^{k-1}}{k^2}\binom{n}{k}=\sum_{j=1}^{n}\frac{1}{j^2}\sum_{k=j}^{n}\frac{1}{k}=\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j^2}. \tag{3}$$

Induct on $n\ge1$. The case $n=1$ is evident. Assume (3) holds for n. For $n+1$, use Pascal's identity:

$$\sum_{j=1}^{n+1}\frac{1}{j}\sum_{k=j}^{n+1}\frac{(-1)^{k-1}\binom{n}{k}}{k^2}+\sum_{j=1}^{n+1}\frac{1}{j}\sum_{k=j}^{n+1}\frac{(-1)^{k-1}\binom{n}{k-1}}{k^2}=\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j^2}+\frac{1}{n+1}\sum_{j=1}^{n+1}\frac{1}{j}\sum_{k=j}^{n+1}\frac{(-1)^{k-1}\binom{n+1}{k}}{k}.$$

Recalling $\sum_{k=\mathbf{1}}^{n}(-1)^{k-\ell}\binom{k-1}{\ell-1}\binom{n}{k}=1$ assures the partial fraction $\sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k\binom{x+k}{k}}=\sum_{j=1}^{n}\frac{1}{x+j}$. Differentiating $\frac{d}{dx}$ the latter at $x=0$ gives $\sum_{k=1}^{n}\frac{(-1)^{k-1}\binom{n}{k}}{k}\sum_{j=1}^{k}\frac{1}{j}=\sum_{j=1}^{n}\frac{1}{j^2}$. Therefore,

$$\frac{1}{n+1}\sum_{j=1}^{n+1}\frac{1}{j}\sum_{k=j}^{n+1}\frac{(-1)^{k-1}\binom{n+1}{k}}{k}=\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{(-1)^{k-1}\binom{n+1}{k}}{k}\sum_{j=1}^{k}\frac{1}{j}=\frac{1}{n+1}\sum_{j=1}^{n+1}\frac{1}{j^2}.$$

Combining the above completes the induction process for (3) and the required proof follows. $\square$

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